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Author(s): Thomas J. Sargent

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Thomas J. Sargent

University of Minnesota and Federal Reserve Bank of Minneapolis

This paper explores some of the implications for econometric practice of the principle that people's observed behavior will change when their constraints change. In dynamic contexts, a proper definition of people's constraints includes among them laws of motion that describe the evolution of the taxes they must pay and the prices of the goods that they buy and sell. Changes in agents' perceptions of these laws of motion (or constraints) will in general produce changes in the schedules that describe the choices they make as a function of the information that they possess. Until very recently, received dynamic econometric practice ignored this principle. The practice of dynamic econometrics should be changed so that it is consistent with the principle that people's rules of choice are influenced by their constraints. This is a substantial undertaking and involves major adjustments in the ways that we formulate, estimate, and simulate econometric models.

Introduction

This paper explores some of the implications for econometric practice of a single principle from economic theory. This principle is that people's observed behavior will change when their constraints change. In dynamic contexts, a proper definition of people's constraints in-

The views expressed here are solely mine and do not necessarily represent the views of the Federal Reserve Bank of Minneapolis or the Federal Reserve System. Many of my thoughts on the subject of this paper have been heavily influenced by numerous discussions with Lars Peter Hansen and Robert E. Lucas, Jr. The observations on Bayesian methods are in large part those of Hansen. Ian Bain made many helpful comments on an earlier draft. This paper is the text for the Mary Elizabeth Morgan Prize lecture, given at the University of Chicago in November 1979.

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cludes among them laws of motion that describe the evolution of the taxes they must pay and the prices of the goods that they buy and sell. Changes in agents' perceptions of these laws of motion (or constraints) will in general produce changes in the schedules that describe the choices they make as a function of the information that they possess. Until very recently, received dynamic econometric practice ignored this principle and routinely deduced policy conclusions by assuming that people's rules of choice would not vary, for example, with the government's choices of laws of motion for variables such as tax rates, government purchases, and so on. These variables are supposed to have their effects precisely because they influence the constraints of some private agents.

The practice of dynamic econometrics should be changed so that it is consistent with the principle that people's rules of choice are influenced by their constraints. This is a substantial undertaking and involves major adjustments in the ways that we formulate, estimate, and simulate econometric models. Foremost, we need a stricter definition of the class of parameters that can be regarded as "structural." The body of doctrine associated with the "simultaneous equations" model in econometrics properly directs the attention of the researcher beyond reduced-form parameters to the parameters of "structural equations," which presumably describe those aspects of the behavior of people that prevail across a range of hypothetical environments. Estimates of the parameters of structural equations are needed in order to analyze an interesting class of policy interventions. Most often, however, included in a prominent way among the "structural equations" have been equations describing the rules of choice for private agents. Consumption functions, investment schedules, demand functions for assets, and agricultural supply functions are all examples of such rules of choice. In dynamic settings, regarding the parameters of these rules of choice as structural or invariant under interventions violates our simple principle from economic theory.

This paper describes methods for interpreting economic time series in a manner consistent with the principle that people's constraints influence their behavior. For the most part, I shall restrict things so that the dynamic economic theory is of the equilibrium variety, with optimizing agents and cleared markets. However, many of the principles described here will pertain to other types of dynamic economic theories, such as "disequilibrium" models with optimizing agents. The line of work I shall describe has diverse antecedents, of which major ones are contributions of Muth (1960, 1961), Nerlove (1967), Lucas and Prescott (1971), Telser and Graves (1971), and Lucas (1972*b*, 1976).[1] The works of Granger (1969) and Sims (1972) have provided key technical econometric foundations.

The basic idea is to interpret a collection of economic time series as resulting from the choices of private agents interacting in markets assumed to be organized along well-specified lines. The private agents are assumed to face nontrivial dynamic and stochastic optimization problems. This is an attractive assumption because the solutions of such problems are known to imply that the chosen variables (e.g., stocks of factors of production or financial assets) can exhibit serial correlation and cross-serial correlation. Since time series of economic data usually have the properties of high own-serial correlation and various patterns of cross-serial correlation, it seems that there is potential for specifying dynamic preferences, technologies, constraints, and rules of the market game that roughly reproduce the serial correlation and cross-correlation patterns in a given collection of time series measuring market outcomes. If this can be done in such a fashion that the free parameters of preferences, technologies, and constraints are identifiable econometrically, it is then possible to interpret the collection of time series as the outcome of a well-specified dynamic, stochastic equilibrium model. This paper is intended as a nontechnical summary of some of the econometric and theoretical issues involved in interpreting data in this way.

But why should anybody want to interpret time-series data as representing the results of interactions of private agents' optimizing choices? The answer is not that this way of modeling is aesthetically pleasing, although it is, nor that modeling in this way guarantees an analysis that implies no role for government intervention, which it does not. The reason for interpreting time series in this way is practical: potentially it offers the analyst the ability to predict how agents' behavior and the random behavior of market-determined variables will each change when there are policy interventions or other changes in the environment that alter some of the agents' dynamic constraints. There is a general presumption that private agents' behavior and the random behavior of market outcomes both will change whenever agents' constraints change, as when policy interventions or other changes in the environment occur. The most that can be hoped for is that the parameters of agents' preferences and technologies will not change in the face of such changes in the environment. If the dynamic

[1] Examples of work in the general line are Holt et al. (1960), Craine (1975), Crawford (1975), Geweke (1977), Sargent (1977, 1978), Blanco (1978), Hansen and Sargent (1979, 1980*a*, 1980*b*), Taylor (1979, 1980), Huntzinger (1979), Kennan (1979), Meese (1979), and Nerlove, Grether, and Carvalho (1979). The philosophy of this work is reviewed by Lucas and Sargent (1978, 1980).

econometric model is formulated explicitly in terms of the parameters of preferences, technologies, and constraints, it will in principle be possible for the analyst to predict the effects on observed behavior of changes in the stochastic environment.

Past dynamic econometric studies should usually be regarded as having been directed at providing ways of summarizing the observed behavior of interrelated variables, without attempting to infer the objectives, opportunities, and constraints of the agents whose decisions determine those variables. Most existing studies can be viewed, at best, as having estimated parameters of agents' decision rules for setting chosen variables as functions of the information they possess. Most of the better studies of consumption, investment, asset demand, and agricultural supply functions must be interpreted as having estimated such decision rules. Dynamic economic theory implies that these decision rules cannot be expected to remain invariant in the face of policy interventions that take the form of changes in some of the constraints facing agents. This means that there is a theoretical presumption that historical econometric estimates of such decision rules will provide poor predictions about behavior in a hypothetically new environment. This was Lucas's (1976) critique of econometric policy evaluation procedures as they existed in 1973.

Some readers of Lucas (1976) have interpreted the message as a call to evaluate policies by using existing econometric models differently.[2] However, one implication of Lucas's argument, and of dynamic economic theory generally, is that the formulation, identification, and estimation of the models must each be approached in substantially new and different ways. Most existing models simply cannot be saved by simulating them a little more shrewdly.[3]

[2] The papers by Anderson (1979) and Mishkin (1979) seem at least partly motivated by this interpretation.

[3] The set of ideas I discuss in this paper has perhaps received most notoriety in the context of macroeconomic examples. In particular, substantial attention has been devoted to the sample economies of Lucas (1972*a*) and Sargent and Wallace (1975) in which those systematic nonneutralities that come from imputing persistently suboptimal expectations to agents were shown to disappear when the hypothesis of rational expectations was imposed on agents. Crudely put, certain classes of systematic monetary policies, in particular those which operate solely via deception, were rendered impotent in the Lucas and Sargent and Wallace examples. Since the publication of these papers, many papers have been published that have described setups in which the choice of systematic policy matters, even when rational expectations prevail. These papers usually invoke a source of nonneutrality not based on deception, of which there are many in standard macroeconomic theory. Papers of this class have often been interpreted as providing a defense of "pre–rational expectations" activist policies along lines that were produced by calculating optimal controls for Keynesian econometric models of the style of the late 1960s. In fact, no such defense is implied, partly because the methods by which optimal controls for government policy variables are calculated are very different in all rational expectations models from the procedures that were applied to pre–rational expectations models, but also because the ways in which

Formulating and estimating "rational expectations" models and dynamic equilibrium models of economic time series involves a variety of important conceptual and econometric issues, some of which I try to summarize in this paper. Among the issues to be treated are the following:

i) *Identification criteria.*—Prior identifying information of the Cowles Commission variety, that is, mainly exclusion restrictions, plays a much smaller role in dynamic equilibrium models. Nonlinear cross-equation restrictions implied by dynamic theory are used extensively. This shift involves important modifications of past ways of thinking about identification and estimation.

ii) *Models of error terms.*—The dynamic equilibrium modeling strategy virtually forces the researcher to think about the sources and interpretations of the error terms in the stochastic equations that he fits. The explicitly stochastic nature of the theorizing makes it difficult to "tack on" error terms after the theorizing is done, a usual procedure in the past.

iii) *The role of Granger causality.*—Granger causality turns out to be a critical concept in the formulation of dynamic economic models, as it is coincident with the condition for the appearance as an information variable in an agent's decision rule of a variable not otherwise in the agent's criterion function or constraints.

iv) *Bayesian analysis.*—Bayesian econometric techniques provide a means of mixing prior theoretical information about parameters with information from the data. Such procedures are widely used by applied time-series econometricians, although often no formal Bayesian justification is given. Dynamic economic theory provides no justification for one widely imposed class of prior restrictions which can be viewed as restrictions directly on decision rules. Instead, dynamic economic theory suggests that prior information about agents' criterion functions and constraints is what should be used in estimation. This feature of dynamic economic theory has implications for the proper implementation both of formal Bayesian procedures and of less formal procedures for constraining parameter estimates.

I shall organize my discussion around an example, namely, a

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econometric estimates are to be constructed for rational expectations models, with or without neutralities, differ substantially from the methods applied to the Keynesian models of the 1960s. The main point of the Lucas (1972*a*) and Sargent and Wallace (1975) examples is that substituting the assumption of rational expectations for "adaptive" expectations makes a critical difference for the methods both by which we should evaluate and optimally choose government policies. That same message is present in the papers of Fischer (1977), Phelps and Taylor (1977), and Hall (1978), even if superficially the differences in some qualitative features of the optimal policies under the two assumptions on expectations may have seemed less dramatic than in Sargent and Wallace's example or Lucas's.

linear-quadratic version of Lucas and Prescott's (1971) model of investment under uncertainty. I shall use this example for discussing the econometric implications of dynamic equilibrium models. I have adopted a linear-quadratic setup because it simplifies both the theoretical and econometric discussions, while illustrating many of the salient methodological implications of dynamic decision theory. Linear-quadratic optimum problems deliver difference equations that are linear in the variables and so match up nicely with much existing dynamic econometric theory. The reader familiar with Lucas and Prescott (1971) will recognize how the example can be generalized to incorporate more general specifications for the technologies, preferences, and constraints. That increased generality would make the econometric implications harder to extract than with the present setup, without altering the basic message.[4]

Investment under Uncertainty

This paper describes a linear-quadratic version of Lucas and Prescott's model of investment and uses it as a vehicle for expositing a variety of conceptual and econometric issues. The model describes the mutual determination over time of the capital stock, output, and market price of a single industry. The model can be generalized to handle multiple factors of production at the cost of what are really only technical complications. Similarly, the model could also be generalized to incorporate a set of industries, like the corn and hog industries, with interacting dynamics. Finally, I mention that it is straightforward to modify the model to incorporate much richer dynamics by generalizing the nature of the adjustment costs.

[4] Using the methods of discounted dynamic programming (e.g., Blackwell 1965), theoretical results establishing existence and uniqueness of equilibria and various qualitative features of the equilibria can often be obtained for "weak" or "general" assumptions, such as that utility is concave, constraint sets are convex and monotone in shift variables, and so on. Lucas and Prescott (1971) and Lucas (1978) give interesting illustrations of these methods. These techniques were also used by Sargent (1980*b*) to make some general observations on interpreting time-series correlations between Tobin's q variable and the aggregate rate of investment. However, for applied work, it is necessary to be able to calculate equilibria as a function of the free parameters of preferences and constraints, and it is highly desirable if the equilibria can be calculated easily. While for general functional forms it is in principle possible to calculate equilibria of recursive competitive models using a contraction mapping, in practice such methods are presently too expensive to use in empirical work. For this reason, for empirical work it is presently necessary to choose functional forms for which equilibria can be calculated either analytically or very quickly. Linear-quadratic specifications are one of the few such choices of convenient functional forms available. (Various versions of logarithmic specification are also sometimes tractable, e.g., Merton [1971].) A valuable treatment of recursive competitive equilibrium models with general specifications of functional forms is Prescott and Mehra (1980).

I define the following variables:

y_t = output of the representative firm;

n = number of firms in the industry, assumed constant over time;

Y_t = ny_t = total output of industry;

P_t = price of output;

D_{1t} = a $(p_1 \times 1)$ vector of random variables appearing in the industry demand schedule, $p_1 \geqslant 1$;

D_{2t} = a $(p - p_1) \times 1$ vector of random variables which help predict future values of the collection of variables D_{1t}, $p \geqslant p_1$;

$D_t = \begin{bmatrix} D_{1t} \\ D_{2t} \end{bmatrix}$;

w_t = rental rate on capital;

W_t = a $(q \times 1)$ vector whose first element is w_t; the remaining elements of W_t are variables that help predict future w_t's;

u_t = a random shock to demand;

ϵ_t = a random shock in the production function;

k_t = stock of capital of the representative firm; and

K_t = nk_t = total capital stock in industry.

The subscript t indexes the date to which the variable corresponds.

I further define the following polynomials in the lag operator L:

$$\delta_u(L) = 1 - \sum_{j=1}^{r_u} \delta_{uj} L^j,$$

where δ_{uj} is a scalar;

$$\delta_D(L) = I_p - \sum_{j=1}^{r_D} \delta_{Dj} L^j,$$

where δ_{Dj} is a $p \times p$ matrix and I_p is the $p \times p$ identity matrix;

$$\delta_w(L) = I_q - \sum_{j=1}^{r_w} \delta_{wj} L^j,$$

where δ_{wj} is a $(q \times q)$ matrix and I_q is the $(q \times q)$ identity matrix; and

$$\delta_\epsilon(L) = 1 - \sum_{j=1}^{r_\epsilon} \delta_{\epsilon j} L^j,$$

where $\delta_{\epsilon j}$ is a scalar.[5]

The industry consists of n identical competitive firms, each of which uses a single factor of production, capital, to produce a single output.

[5] I shall impose the condition that the zeroes of $\delta_\epsilon(z)$, $\delta_u(z)$, det $\delta_D(z)$ and det $\delta_w(z)$ each exceed unity in modulus. Actually, a weaker condition would suffice, namely, that the zeroes of these polynomials each exceed $\sqrt{\beta}$ in modulus, where β is the discount factor introduced below. These conditions on the zeroes are regularity conditions that assure that the infinite series calculated in eqq. (14) and (19) converge.

Output of the representative firm y_t is governed by

$$y_t = fk_t + n^{-1}\epsilon_t, \qquad f > 0, \tag{1}$$

where k_t is the representative firm's stock of capital at t, and ϵ_t is a random error in the technology. The firm knows $\{\epsilon_t, \epsilon_{t-1}, \ldots\}$, but does not know with certainty future values of the shock ϵ_t. The error ϵ_t is known to follow the r_ϵth-order Markov process

$$\delta_\epsilon(L)\epsilon_t = V_t^\epsilon, \tag{2}$$

where V_t^ϵ is a "fundamental" white-noise error term for ϵ_t.[6] The firm is assumed to know $\delta_\epsilon(L)$ and $E(V_t^\epsilon)^2$ with certainty.

The demand curve for output is given by[7]

$$P_t = A_0 - A_1 Y_t + A_2 D_{1t} + u_t, \qquad A_0, A_1 > 0, \tag{3}$$

where D_{1t} is a $(p_1 \times 1)$ vector of "demand shifters," A_2 is a $(1 \times p_1)$ vector of constants, and u_t is a random shock to the demand curve. The random term u_t obeys the r_uth-order Markov process

$$\delta_u(L)u_t = V_t^u, \tag{4}$$

where V_t^u is a fundamental white noise for u_t. The $(p_1 \times 1)$ vector of demand shifters D_{1t} consists of the first p_1 rows of the $p \times 1$ vector D_t, which follows the r_Dth-order vector autoregressive process

$$\delta_D(L)D_t = V_t^D, \tag{5}$$

where V_t^D is a $(p \times 1)$ vector white noise that is fundamental for the process D_t. The representative firm is assumed to know $\delta_u(L)$, $\delta_D(L)$, A_0, A_1, A_2, and the second moments of V_t^u and V_t^D with certainty.

At time t, total output is given by

$$Y_t = ny_t = fK_t + \epsilon_t. \tag{6}$$

The representative firm's problem is to choose a contingency plan for k_{t+j} to maximize the criterion

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[P_t y_t - w_t k_t - \frac{d}{2}(k_{t+1} - k_t)^2\right], \tag{7}$$

[6] An $(n \times 1)$ vector white noise v_t^x is said to be fundamental for an $(n \times 1)$ vector process x_t if the vector of one-step-ahead linear least-squares errors in predicting x_t from past x's can be written as a linear combination of n components of v_t^x.

[7] Since a simple static demand function is posited, all of the interesting dynamics of the model come from its supply side. Specifying a demand schedule with interesting dynamics would complicate the presentation but not alter the basic messages of our example. Telser and Graves (1971) analyze dynamic optimization problems in which much of the interesting dynamics come from a demand curve that is specified. Sargent (1979, chap. 16) analyzes a model of the labor market in which the dynamics are influenced by nontrivial dynamic optimization problems solved by both suppliers and demanders.

subject to k_0 given. In (7), E_t is the mathematical expectation operator, conditional on information known to the firm at time t. This information set will shortly be specified precisely. In (7), d is a positive constant. The term $(d/2)(k_{t+1} - k_t)^2$ is intended to represent the notion that there are costs internal to the firm of adjusting the capital stock and that these rise at an increasing rate with the absolute value of the change in capital. We assume that the rental on capital w_t is the first element of the $(q \times 1)$ vector random process W_t that obeys the r_wth-order vector autoregression

$$\delta_w(L)W_t = V_t^w, \tag{8}$$

where V_t^w is a $(q \times 1)$ vector white noise that is fundamental for W_t. The firm is supposed to know $\delta_w(L)$ and the second-moment matrix of V_t^w with certainty.

At time t, the firm chooses k_{t+1}, given the information that it has available at t. However, the maximization problem (7) is not yet well posed, since we have not completely spelled out the dynamic constraints with respect to which the maximization is supposed to occur. To complete the problem (7), we begin by substituting $(fk_t + n^{-1}\epsilon_t)$ for y_t, and $(A_0 - A_1 fK_t - A_1\epsilon_t + A_2 D_{1t} + u_t)$ for P_t to get

$$E_0 \sum_{t=0}^{\infty} \beta^t \Big[(A_0 - A_1 fK_t - A_1\epsilon_t + A_2 D_{1t} + u_t)(fk_t + n^{-1}\epsilon_t) - w_t k_t - \frac{d}{2}(k_{t+1} - k_t)^2\Big]. \tag{9}$$

In order that the problem of maximizing (9) with respect to a contingency plan for $\{k_{t+j}\}$ be well posed, it is necessary to attribute to the firm precise views about the laws of motion of the random variables that it cannot control, but whose values influence the best choice of its own stocks of capital. For problem (9), these uncontrollable variables about which the representative firm cares are K_t, D_{1t}, u_t, ϵ_t, and w_t. The firm cares about the present and future behavior of the variables $(K_t, D_{1t}, u_t, \epsilon_t)$ because they influence the present and future behavior of the market price through the demand relationship $P_t = A_0 - A_1 fK_t - A_1\epsilon_t + A_2 D_{1t} + u_t$. The firm cares about the evolution of the rental process w_t because it influences its costs. We have already completely described our assumptions about the firm's views of the laws of motion of D_{1t}, u_t, ϵ_t, and w_t, namely, that the firm knows the Markov laws (4), (5), (2), and (8) that govern them, and at time t knows $D_t, D_{t-1}, \ldots, u_t, u_{t-1}, \ldots, \epsilon_t, \epsilon_{t-1}, \ldots$, and $W_t, W_{t-1}, \ldots$. To complete the specification requires that we specify the firm's views about the evolution of the aggregate capital stock K_t. We assume that the representative firm believes that the aggregate capital stock evolves according to the law

$$K_{t+1} = H_0 + H_w(L)W_t + H_D(L)D_t + H_\epsilon(L)\epsilon_t + H_u(L)u_t + H_1K_t, \tag{10}$$

where H_0 and H_1 are scalars and

$$H_w(L) = \sum_{j=0}^{r_w-1} H_{wj}L^j, \qquad \text{where } H_{wj} \text{ is } (1 \times q);$$

$$H_D(L) = \sum_{j=0}^{r_D-1} H_{Dj}L^j, \qquad \text{where } H_{Dj} \text{ is } (1 \times p);$$

$$H_\epsilon(L) = \sum_{j=0}^{r_\epsilon-1} H_{\epsilon j}L^j, \qquad \text{where } H_{\epsilon j} \text{ is a scalar; and}$$

$$H_u(L) = \sum_{j=0}^{r_u-1} H_{uj}L^j, \qquad \text{where } H_{uj} \text{ is a scalar.}$$

The representative firm is assumed to know all of the parameters of the linear law of motion (10) with certainty. The reason that we have chosen the form (10) as the firm's perceived law of motion for K will shortly become apparent.

With these specifications, the maximization of (9) is now well posed. Summarizing the setup, we have that the representative firm maximizes

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[(A_0 - A_1 fK_t - A_1\epsilon_t + A_2D_{1t} + u_t)(fk_t + n^{-1}\epsilon_t) - w_tk_t - \frac{d}{2}(k_{t+1} - k_t)^2\right], \tag{9}$$

subject to the laws of motion[8]

$$K_{t+1} = H_0 + H_w(L)W_t + H_D(L)D_t + H_\epsilon(L)\epsilon_t + H_u(L)u_t + H_1K_t, \tag{10}$$

$$\delta_w(L)W_t = V_t^w, \tag{8}$$

$$\delta_u(L)u_t = V_t^u, \tag{4}$$

$$\delta_D(L)D_t = V_t^D, \tag{5}$$

$$\delta_\epsilon(L)\epsilon_t = V_t^\epsilon, \tag{2}$$

and subject to the information set at time t,[9]

[8] It would be straightforward to modify this setup to assume that the $\{W, u, \epsilon, D\}$ processes are each finite order mixed moving average, autoregressive processes. For the details, see Hansen and Sargent (1979).

[9] These variables completely characterize the "state" vector for the firm's problem.

$$\{K_t, k_t, W_t, W_{t-1}, \ldots, W_{t-r_w+1}, D_t, D_{t-1}, \ldots, D_{t-r_D+1}, \epsilon_t, \epsilon_{t-1}, \ldots, \epsilon_{t-r_\epsilon+1}, u_t, u_{t-1}, \ldots, u_{t-r_u+1}\}.$$

The firm maximizes (9), taking the laws of motion (8), (4), (5), (2), and (10) as given and beyond its control. The firm is assumed to behave competitively and to act as if it has no control over the aggregate capital stock K. This is a reasonable assumption if n is large. The firm is assumed to know the δ's and H's with certainty and to know the first and second moments of the V_t's.[10] We further restrict the problem so that the solution is a linear contingency plan.[11] For this to be true, it is sufficient that the least-squares predictors of future W, D, ϵ, and u's be linear in the conditioning variables. This will be true if V_t^ϵ, V_t^u, V_t^D, and V_t^w obey normal probability laws. Alternatively, the analyst can simply assume that the industry is operating under optimal linear rules. In either case, the solution of the representative firm's problem is a linear contingency plan of the form[12]

$$k_{t+1} = h_0 + h_w(L)W_t + h_D(L)D_t + h_\epsilon(L)\epsilon_t + h_u(L)u_t + h_1K_t + h_2k_t, \tag{11}$$

where h_0, h_1, and h_2 are scalars, and

$$h_w(L) = \sum_{j=0}^{r_w-1} h_{wj}L^j, \qquad \text{where } h_{wj} \text{ is } (1 \times q);$$

$$h_D(L) = \sum_{j=0}^{r_D-1} h_{Dj}L^j, \qquad \text{where } h_{Dj} \text{ is } (1 \times p);$$

$$h_\epsilon(L) = \sum_{j=0}^{r_\epsilon-1} h_{\epsilon j}L^j, \qquad \text{where } h_{\epsilon j} \text{ is a scalar; and}$$

$$h_u(L) = \sum_{j=0}^{r_u-1} h_{uj}L^j, \qquad \text{where } h_{uj} \text{ is a scalar.}$$

The h's of (11) are in general functions both of the parameters in the criterion function (9), the parameters of δ_w, δ_u, δ_D, and δ_ϵ appearing in

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We have in mind that the firm actually has observations on values of K, k, W, D, ϵ, and u for all dates t and earlier. It turns out that the firm's decisions are optimally a function only of the information set listed in the text.

[10] It is assumed that each of V_t^w, V_t^u, V_t^D, and V_t^ϵ is orthogonal to the information set $\{W_{t-s}, u_{t-s}, D_{t-s}, \epsilon_{t-s}, s \geq 1\}$.

[11] This is because we want the stochastic difference equations describing the behavior of the system to be linear and thereby to be readily susceptible to econometric analysis.

[12] That the solution to the problem is of this form follows from linear optimal control theory (see Kushner 1971, chap. 9; Bertsekas 1976, chap. 3; Sargent 1979, chap. 14; or Kwakernaak and Sivan 1972).

(8), (4), (5), and (2), and the H's of the perceived law of motion for capital (10). The mapping giving the h's as functions of these other parameters is defined implicitly by standard formulas in linear optimal control theory, as exposited, for example, by Kwakernaak and Sivan (1972) and Bertsekas (1976). For present purposes, it is enough to note the existence of this mapping without exploring its nature in detail. The economic content of the mapping from the δ's, H's, and objective function parameters to the h parameters of the firm's decision rule is easy to understand, since it captures the notion that the firm's rule of choice depends on both its objective and its perceived constraints (10), (8), (4), (5), and (2).

Multiplying both sides of the firm's decision rule (11) by n and using $K_t = nk_t$ gives

$$\begin{aligned} K_{t+1} = nh_0 + nh_w(L)W_t + nh_D(L)D_t + nh_\epsilon(L)\epsilon_t \\ + nh_u(L)u_t + (nh_1 + h_2)K_t. \end{aligned} \tag{12}$$

Equation (12) is the actual law of motion for aggregate capital that results from the behavior of the representative firm. The representative firm's optimization problem in effect induces a mapping from the firm's perceived law of motion for aggregate capital (10) to the actual law of motion (12). For each possible particular perceived law of motion of the form (10), there is an implied law of motion for aggregate capital of the form (12). The notion of rational expectations is that the representative firm's perceptions of (10) are correct. In effect, a rational expectations equilibrium is a fixed point of the mapping that the representative firm's optimization problem induces from (10) to (12). Formally, we define a *rational expectations equilibrium* as a perceived law of motion (10) and an implied actual law of motion (12) which are identically equal. In a rational expectations equilibrium, firms' perceptions about the law of motion for aggregate capital turn out to be confirmed by the aggregate of the choices made by firms. Upon comparing (10) with (12) it is evident that necessary and sufficient conditions for a rational expectations equilibrium are

$$\begin{aligned} H_0 &= nh_0, \\ H_w(L) &= nh_w(L), \\ H_D(L) &= nh_D(L), \\ H_\epsilon(L) &= nh_\epsilon(L), \\ H_u(L) &= nh_u(L), \\ H_1 &= (nh_1 + h_2). \end{aligned}$$

Implicit in the above definition of a rational expectations equilibrium are the following elements: (*a*) market clearing, (*b*) optimization of the firm's expected present value, and (*c*) correct perceptions on the part of firms of the laws of motion of variables affecting their present value but beyond their control.

We begin our analysis of the model by briefly describing aspects of the optimization problem solved by the firm. Among the first-order necessary conditions for the maximization of (9) is the following system of stochastic "Euler equations," which are derived by differentiating (9) with respect to k_t for $t = 1, 2, \ldots$:

$$\beta d k_{t+1} - d(1+\beta)k_t + dk_{t-1} = \beta w_t - \beta f(A_0 - A_1 f K_t - A_1 \epsilon_t + A_2 D_{1t} + u_t), \tag{13}$$

or

$$k_{t+1} - \left(\frac{1}{\beta} + 1\right)k_t + \frac{1}{\beta}k_{t-1} = \frac{1}{d}w_t - \frac{f}{d}P_t.$$

In addition to the system of Euler equations, a transversality condition is among the first-order necessary conditions. The transversality condition can be derived by methods described in Sargent (1979). The transversality condition for the present problem in effect requires that the solution possess the property

$$\lim_{j\to\infty} E_t \beta^{t+j} k_{t+j} = 0.$$

Using the lag operator, the preceding Euler equation can be rewritten as[13]

$$\left[1 - \left(\frac{1}{\beta} + 1\right)L + \frac{1}{\beta}L^2\right]k_{t+1} = \frac{1}{d}w_t - \frac{f}{d}P_t.$$

Using the factorization

$$\left[1 - \left(\frac{1}{\beta} + 1\right)L + \frac{1}{\beta}L^2\right] = \left(1 - \frac{1}{\beta}L\right)(1 - L),$$

the above Euler equation can be written as

$$\left(1 - \frac{1}{\beta}L\right)(1 - L)k_{t+1} = \frac{1}{d}w_t - \frac{f}{d}P_t.$$

Noting that $[1 - (1/\beta)L] = -\beta^{-1}L(1 - \beta L^{-1})$ and operating on both sides of the above equation with $[-\beta^{-1}L(1 - \beta L^{-1})]^{-1}$ gives the solution[14]

[13] For a discussion of the use of lag operators in the present context, see Sargent (1979, chaps. 9 and 14).

[14] In effect, the transversality condition compels us to solve the unstable root forward in this manner.

$$(1 - L)k_{t+1} = \frac{-d^{-1}\beta L^{-1}}{1 - \beta L^{-1}} w_t + \frac{\beta f d^{-1} L^{-1}}{1 - \beta L^{-1}} P_t,$$

or, equivalently,

$$(1 - L)k_{t+1} = -d^{-1}\beta \sum_{i=0}^{\infty} \beta^i w_{t+i+1} + \beta f d^{-1} \sum_{i=0}^{\infty} \beta^i P_{t+i+1}. \quad (14)$$

It can be verified that (14) satisfies both the Euler equations and the transversality condition. Equation (14) would give the appropriate rule for setting k_{t+1} if the firm had perfect foresight about the entire future paths of the rental w_t and the output price P_t. When the firm does not have perfect foresight, the correct decision rule can be derived by replacing the future values on the right side of (14) with the corresponding mathematical expectations conditional on information the firm does have. This leads to the decision rule[15]

$$(1 - L)k_{t+1} = -d^{-1}\beta \sum_{i=0}^{\infty} \beta^i E w_{t+i+1} \mid \Omega_t + \beta f d^{-1} \sum_{i=0}^{\infty} \beta^i E P_{t+i+1} \mid \Omega_t. \quad (15)$$

Here Ω_t is defined as the information set $\Omega_t = \{W_t, W_{t-1}, \ldots, u_t, u_{t-1}, \ldots, D_t, D_{t-1}, \ldots, \epsilon_t, \epsilon_{t-1}, \ldots, K_t\}$. The conditional mathematical expectations are assumed to be computed using the laws of motion (10), (8), (4), (5), and (2) for K, W, u, D, and ϵ, respectively, as well as the demand relationship $P_t = A_0 - A_1(fK_t + \epsilon_t) + A_2 D_{1t} + u_t$, which is used to deduce the law of motion for P_t. Once these conditional mathematical expectations are explicitly calculated in terms of the parameters of (10), (8), (4), (5), (2), and the demand curve (3), they can be substituted into equation (15) to deduce the optimum decision rule (11) for the representative firm. The decision rule (11) is linear in all of the information variables that appear on the right side. However, as the above method of calculating the parameters h of the decision rule (11) suggests, the parameters h are themselves complicated nonlinear functions of the underlying parameters of the model: the parameters A_0, A_1, A_2 of the demand curve, the parameters f and d of the technology, and the parameters $\delta_u(L)$, $\delta_\epsilon(L)$, $\delta_w(L)$, and $\delta_D(L)$ of the laws of motion of the random processes given from outside the model.[16] The h's are also nonlinear functions of the H's of the law of

[15] As noted above, we shall want the relevant conditional expectations to be linear. So we shall regard the $E(\cdot \mid \Omega_t)$ that appears in (15) and elsewhere as wide-sense conditional expectations, that is, linear least-squares predictors. This amounts to restricting the firm to linear decision rules, as desired.

[16] The parameters β and n also belong in this list of underlying parameters of the model. I shall usually delete these two parameters from subsequent listings of the model's underlying parameters, though they should be understood. In some applications, the analyst may want to specify counterparts of β and n completely a priori, in which case they would not be included among the free parameters of the model over which the likelihood function or other measure of "fit" is to be maximized.

motion of aggregate capital (10), which are not given from outside but are to be determined from the analysis. The nature of these nonlinearities has been characterized by Hansen and Sargent (1980*b*) and will be alluded to further below.

Equation (15), which was derived by purely formal manipulations, has the virtue of indicating clearly that the firm has an incentive to forecast future realizations of the rental w and the output price P. As a result, any state variables that the firm sees at t, and that help predict either future P's or future w's, will appear in the firm's decision rule for k_{t+1}, given by equation (11). That the h's of (11) are nonlinear functions of the parameters $\{A_0, A_1, A_2, f, d, \beta, \delta_u, \delta_\epsilon, \delta_w, \delta_D, H_u, H_\epsilon, H_w, H_D, H_0$, and $H_1\}$ stems from the nonlinear way in which the conditional mathematical expectations of future w's and P's are functions of these parameters.

In practice, to compute a rational expectations equilibrium it is not necessary ever to calculate the right side of (15). Indeed, it is never necessary explicitly to calculate the h's that determine the decision rule (11) of the representative firm. Instead, the H's of the equilibrium law of motion for the industry can be calculated directly as follows.[17] First, multiply both sides of equation (13) by n, then use $K_t = nk_t$ and collect all terms in K on the left side to get

$$\begin{aligned}\beta dK_{t+1} - [d(1+\beta) + A_1 f^2 \beta n]K_t + dK_{t-1} = \beta n w_t \\ - \beta n f A_0 + A_1 \beta f n \epsilon_t - \beta f n A_2 D_{1t} - \beta f n u_t.\end{aligned} \tag{16}$$

It is of some interest that (16) is itself the Euler equation for the "social planning" problem of maximizing[18]

$$\begin{aligned} E_0 \sum_{t=0}^{\infty} \beta^t \Big\{ \Big[A_0(fK_t + \epsilon_t) - \frac{1}{2} A_1 (fK_t + \epsilon_t)^2 + (fK_t + \epsilon_t) A_2 D_{1t} \\ + (fK_t + \epsilon_t) u_t \Big] - w_t K_t - \frac{1}{2} n^{-1} d (K_{t+1} - K_t)^2 \Big\}, \end{aligned} \tag{17}$$

[17] The following argument in the text provides a way of discovering Lucas and Prescott's (1971) method of calculating the rational expectations equilibrium by formulating a fictitious social planning problem that reproduces the equilibrium. It is worth remarking that Kydland and Prescott (1977) describe a recursive method of calculating a linear rational expectations equilibrium that is applicable to our problem and is distinct from the Lucas-Prescott method upon which the discussion in the text is based. Kydland and Prescott's method successfully computes the equilibrium even in instances in which the Lucas-Prescott method breaks down. These instances occur, e.g., in which there is feedback from the industry-wide aggregate capital stock K to W or D, as would occur if lagged K's appeared as states in the Markov law for W or D. In such instances, Lucas and Prescott's social planning problem fails to reproduce the rational expectations equilibrium essentially because the fictitious planner takes into account the externality that the feedback from K to W or D constitutes.

[18] This was emphasized in a more general context by Lucas and Prescott (1971).

subject to the laws of motion (8), (4), (5), and (2) for w_t, u_t, D_{1t}, and ϵ_t[19] and subject to K_0 given.

The term in brackets is the area under the demand curve, since

$$\int_0^{Y_t} (A_0 - A_1 x + A_2 D_{1t} + u_t) dx = A_0 Y_t - \frac{1}{2} A_1 Y_t^2 + Y_t A_2 D_{1t} + Y_t u_t.$$

Thus (17) is the discounted area under the demand curve minus the total costs of production. Dividing each side of (16) by βd, the Euler equation can be written

$$\begin{aligned} K_{t+1} - \left(1 + \frac{1}{\beta} + \frac{A_1 f^2 n}{d}\right) K_t + \frac{1}{\beta} K_{t-1} \\ = \frac{n}{d} w_t - \frac{nfA_0}{d} + \frac{A_1 fn}{d} \epsilon_t \\ - d^{-1} fnA_2 D_{1t} - \frac{fn}{d} u_t. \end{aligned} \tag{18}$$

It can easily be proved that there exists a λ such that

$$[1 - (1 + \beta^{-1} + A_1 f^2 n d^{-1})L + \beta^{-1} L^2] = [1 - (\lambda\beta)^{-1} L](1 - \lambda L),$$

where $|\lambda| < 1/\sqrt{\beta}$.[20] Using $-(\lambda\beta)^{-1} L(1 - \lambda\beta L^{-1}) = [1 - (\lambda\beta)^{-1} L]$, we have that the Euler equation (18) can be written as

$$\begin{aligned} -(\lambda\beta)^{-1} L(1 - \lambda\beta L^{-1})(1 - \lambda L) K_{t+1} = \frac{-nfA_0}{d} + \frac{n}{d} w_t \\ + \frac{A_1 fn}{d} \epsilon_t - \frac{fn}{d} A_2 D_{1t} - \frac{fn}{d} u_t. \end{aligned}$$

A solution of the Euler equation that also satisfies the transversality condition for the social planning problem is

$$\begin{aligned} (1 - \lambda L) K_{t+1} = \frac{+\lambda\beta nfA_0}{d} (1 - \lambda\beta)^{-1} - \frac{n\lambda\beta}{d} \frac{L^{-1}}{1 - \lambda\beta L^{-1}} w_t \\ - \frac{A_1 fn\lambda\beta}{d} \frac{L^{-1}}{1 - \lambda\beta L^{-1}} \epsilon_t + \frac{fn\lambda\beta d^{-1} L^{-1}}{1 - \lambda\beta L^{-1}} A_2 D_{1t} \\ + \frac{fn\lambda\beta d^{-1} L^{-1}}{1 - \lambda\beta L^{-1}} u_t. \end{aligned} \tag{19}$$

Recall, for example, that $(1 - \lambda\beta L^{-1})^{-1} w_t = \sum_{j=0}^{\infty} (\lambda\beta)^j w_{t+j}$. Then it can be recognized that equation (19) is the perfect foresight solution of

[19] It can also be proved that the transversality condition for (17) imposes the same condition on the solution as does the transversality condition of the representative firm.

[20] This follows directly from the observation that if z_0 is a zero of $[1 - (1 + \beta^{-1} + A_1 f^2 n d^{-1})z + \beta^{-1} z^2]$, then so is βz_0^{-1}.

the planning problem that the rational expectations competitive equilibrium implicitly solves. Thus, equation (19) expresses the aggregate capital stock K_{t+1} as a linear function of K_t and all future values of w_t, ϵ_t, D_{1t}, and u_t.

By using the methods of Hansen and Sargent (1980*b*, esp. appendix A), equation (19) can be converted to the "realizable" law for K that satisfies the Euler equations and transversality conditions, and which expresses K_{t+1} as a function only of information known at time t.[21] This involves replacing the terms w_{t+i}, ϵ_{t+i}, D_{1t+i}, and u_{t+i} in (19) by the corresponding mathematical expectations conditioned on Ω_t. The resulting equilibrium law of motion for K can be shown to be

$$\begin{aligned} K_{t+1} = H_0 + H_w(L)W_t + H_D(L)D_t + H_\epsilon(L)\epsilon_t \\ + H_u(L)u_t + H_1 K_t, \end{aligned} \tag{20}$$

where

$$\begin{aligned} H_0 &= \frac{+\lambda\beta nfA_0}{d(1-\lambda\beta)}, \\ H_1 &= \lambda, \\ H_w(L) &= \frac{-n\lambda\beta}{d}\phi_w\left\{\frac{L^{-1}[I-\delta_w(\lambda\beta)^{-1}\delta_w(L)]}{1-\lambda\beta L^{-1}}\right\}, \\ H_\epsilon(L) &= \frac{-A_1 fn\lambda\beta}{d}\left\{\frac{L^{-1}[1-\delta_\epsilon(\lambda\beta)^{-1}\delta_\epsilon(L)]}{1-\lambda\beta L^{-1}}\right\}, \\ H_D(L) &= +fn\lambda\beta d^{-1}A_2\phi_D\left\{\frac{L^{-1}[I-\delta_D(\lambda\beta)^{-1}\delta_D(L)]}{1-\lambda\beta L^{-1}}\right\}, \\ H_u(L) &= +d^{-1}fn\lambda\beta\left\{\frac{L^{-1}[1-\delta_u(\lambda\beta)^{-1}\delta_u(L)]}{1-\lambda\beta L^{-1}}\right\}. \end{aligned} \tag{21}$$

Here ϕ_w is a $1 \times q$ vector with 1 in the first position, followed by $(q - 1)$ zeroes, and ϕ_D is a $p_1 \times p$ matrix with a $(p_1 \times p_1)$ identity matrix as the first p_1 columns and zeroes elsewhere. Notice that $w_t \equiv \phi_w W_t$ and $D_{1t} \equiv \phi_D D_t$. It is convenient at this point to recall the laws of motion assumed for w_t, u_t, ϵ_t, and D_{1t}, namely,

$$\delta_w(L)W_t = V_t^w, \tag{8}$$

$$\delta_u(L)u_t = V_t^u, \tag{4}$$

[21] Note that eq. (19) satisfies the first-order necessary conditions for the optimization problem but gives the planner too much information (it is "anticipative" or "nonrealizable"). The correct solution to the problem taking the information set available to the planner into account is the solution of the first-order necessary conditions that expresses K_{t+1} as a function only of information that the planner possesses at time t. Such a solution is said to be "realizable" or "nonanticipative."

$$\delta_D(L)D_t = V_t^D, \tag{5}$$

$$\delta_\epsilon(L)\epsilon_t = V_t^\epsilon. \tag{2}$$

Equation (20) expresses the equilibrium law for the industry-wide capital stock K_{t+1} as a linear function of K_t and current and past values of W, D, ϵ, and u. Current and past values of W appear in (20) because they help predict future values of the rental rate w_t, while current and past values of D, ϵ, and u appear because they are used by agents to predict the future course of the market price P. The numbers of lagged values of W, D, ϵ, and u in (20) are $r_w - 1$, $r_D - 1$, $r_\epsilon - 1$, and $r_u - 1$, as expressions (21) can be used to show.[22] Thus, the numbers of lagged values of these "information variables" W, D, ϵ, and u in (20) are entirely inherited from the specifications of the actual laws of motion for W, D, ϵ, and u in (8), (5), (2), and (4).

Notice that the appearance of w_t and D_{1t} in the objective function of the representative firm (9) (or equivalently in the objective function of the fictitious social planner [17]) gives rise to the appearance in (20) of the entire blocks of variables W_t and D_t that help predict w and D_{1t}, respectively. Thus any variables that help predict w and D_{1t}, and which agents have information on, belong in the equilibrium law of motion for industry-wide capital. The property that the remaining variables in W (or D) help predict future values of w (or D_1) is said to be the property that the remaining variables in W (or D) *Granger cause* w (or D_1). The notion of Granger causality thus turns out to be coincident with the criterion for whether random variables that do not themselves appear in the agent's criterion function nevertheless end up in the equilibrium law of motion or decision rule, essentially because they appear in the agents' constraints as information variables that help predict variables that do appear in the criterion function. It is mainly for this reason that the concept of Granger causality has played an important role in work with rational expectations models.[23]

[22] By expanding the polynomial in L, it is possible to show that

$$\frac{L^{-1}[I - \delta(\lambda\beta)^{-1}\delta(L)]}{1 - \lambda\beta L^{-1}} = \delta(\lambda\beta)^{-1}\left\{\sum_{j=0}^{r-1}\left[\sum_{k=j+1}^{r}(\lambda\beta)^{k-j-1}\delta_k\right]L^j\right\}, \tag{22}$$

where $\delta(L) = I - \sum_{j=1}^{r}\delta_j L^j$. Notice that the polynomial on the left side of (22) is one sided in nonnegative powers of L, despite the appearance of L^{-1}, and that it is a polynomial of order $(r - 1)$, as asserted in the text. The formula (22) can be derived by mimicking the procedures used in Hansen and Sargent (1980*a*). The same mathematical techniques used by Hansen and Sargent (1980*a*) to derive expressions like (21) or (22) were independently utilized by Futia (1979) to compute linear rational expectations equilibria. Also, without knowing of Hansen and Sargent's work, John Kennan independently derived formulas similar to (22) in a personal letter to me.

[23] From this point of view, it is irrelevant whether Granger causality is consistent with one's notion of what "true" causality is. Sims (1972) has described the relationship of the concept of Granger causality to that of strict econometric exogeneity. That a random process y fail to Granger cause x is a necessary condition for x to be strictly econometri-

Equations (20) and (21) reveal explicitly how the parameters of the equilibrium law of motion for industry-wide capital are themselves nonlinear functions of the underlying parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w(L), \delta_u(L), \delta_\epsilon(L), \delta_D(L)\}$. The nonlinearity has two sources. First, there is the fact that λ is a nonlinear function of β and $(A_1 f^2 n d^{-1})$ via the factorization defining λ, $[1 - (\lambda\beta)^{-1}L](1 - \lambda L) = [1 - (1 + \beta^{-1} + A_1 f^2 n d^{-1})L + \beta^{-1}L^2]$. Second, given λ, the formulas for $H_w(L), H_u(L), H_\epsilon(L)$, and $H_D(L)$ in (21) are nonlinear in the parameters of $\delta_w(L)$, $\delta_u(L)$, $\delta_\epsilon(L)$, and $\delta_D(L)$. Nonlinear cross-equation restrictions of the kind illustrated by (20) and (21) are the hallmark of rational expectations models. Such cross-equation restrictions are largely absent from "pre–rational expectations" dynamic econometric models.[24] The presence of these restrictions impinges on a variety of fundamental econometric and conceptual issues, including identification, the analysis of interventions, models of "error terms," and the role of "prior information." I now turn to discussing each of these issues, using (20) and (21) as an instrument.

Analysis of Interventions

At this point, it is useful to remind ourselves of the principal reason that an economist might want to construct a dynamic econometric model of an industry along the lines of our example. It is to be able to make quantitative predictions about the effects on the industry that various hypothetical interventions or "changes in the environment" will have. In the present context, a hypothetical "intervention" or "change in the environment" means a change in one of the polynomials $\delta_w(L)$, $\delta_u(L)$, $\delta_D(L)$, or $\delta_\epsilon(L)$ that describe, respectively, the stochastic processes for W, u, D, and ϵ that impinge on the market.[25] Several interesting examples of such interventions can be given, including the following:

a) Suppose that there is a specific tax imposed on sales of the

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cally exogenous with respect to y. For this reason, the concept of Granger causality is also useful in designing specification tests. For a discussion of the relationship between Granger causality and econometric exogeneity in the context of linear rational expectations models, see Hansen and Sargent (1980*a*).

[24] Thus Fisher wrote: "In practice, except for such covariance restrictions [across disturbances in distinct structural equations], restrictions which relate the parameters of one equation to those of one or more others are extremely rare. There is no reason in principle why such cases cannot occur, however, and it may be worthwhile devoting a very short discussion to them" (1966, p. 176).

[25] By now, this is a routine and uncontroversial definition of an intervention. Applications of the techniques of optimal control theory to the calculation of macroeconometric and microeconometric policy response functions employ precisely this concept of intervention (see, e.g., Chow 1973; Kareken, Muench, and Wallace 1973; Arzac and Wilkinson 1979; and Taylor 1979).

product. Such a specific tax can be modeled as a component of $(A_2 D_{1t})$. Since the behavior of the tax through time will be described by an element of the vector Markov law $\delta_D(L)D_t = V_t^D$, changes in the rule for setting the specific tax amount to changes in one of the rows of $\delta_D(L)$.

b) Suppose that there is a specific tax on the use of the factor of production. This tax can be modeled as an addition to the rental w_t. A change in the rule for setting this tax can be modeled as a change in a row of $\delta_w(L)$.

c) Suppose there is a change in the structure of the process governing the "pretax" part of the rental. Again this can be modeled as a change in one row of $\delta_w(L)$. With a little imagination, the effects of a change in the organization of the industry[26] supplying the factor might be modeled in this way.

The model leading to (20) and (21) provides a way of predicting quantitatively the effects of such changes, once agents have caught on to them. The effect of interventions in the sense described here is to change the function (20) describing the evolution of industry capital in a way predicted by the formulas given in (21). Since interventions of this class change the law of motion (20), it is necessary to have analytic methods which use the cross-equation restrictions (21) to predict how the H's of the K-law of motion (20) will change if there is a hypothetical intervention operating on one or more of the δ's.

In order to evaluate policy interventions in this way, it is essential that the H's of (20) should not be viewed as being among the free parameters of the model. Instead, the model's free parameters are to be regarded as the deeper parameters $\{A_0, A_1, A_2, f, d, \delta_w, \delta_u, \delta_\epsilon, \delta_D\}$. The researcher needs to know these parameters in order to be able to use the formulas (21) to predict the consequences of hypothetical changes in the functions δ.[27]

From the dynamic economic theory leading to (20) and (21), it is

[26] E.g., if it becomes a cartel when before it had been competitive or noncooperative in some way.

[27] A technical qualification needs to be added at this point. In order to have a model capable of predicting effects of interventions acting on the δ's, one can sometimes get by without having uniquely identified the parameters $\{A, f, d\}$. What the researcher must identify are the parameters of the characteristic polynomial of the Euler equation (16), namely the parameters ϕ_0, ϕ_1 in $\{\beta d - [d(1 + \beta) + A_1 f^2 n\beta]L + dL^2\} = (\beta\phi_1 + \phi_0 L + \phi_1 L^2)$, where $\phi_1 = d$ and $-\phi_0 = [d(1 + \beta) + A_1 f^2 n\beta]$. The theory assumes that the ϕ_j's will be invariant with respect to interventions on the δ's. If the researcher can uniquely identify the ϕ_j's, he can proceed with econometric policy evaluation, even if he cannot uniquely identify all of (A_1, f, d). In some setups, the parameters of the characteristic polynomials of the Euler equations are identified even though only an equivalence class of the counterparts of (A_1, f, d) is identified. This is enough for econometric policy evaluation to proceed. This problem is discussed by Hansen and Sargent (1980*b*). It is technically related to the "inverse optimal control" problem (see Mosca and Zappa 1979).

evident that a given numerical version of (20), estimated from historical data, cannot be used to evaluate the consequences of arbitrary input sequences for $\{W_t\}$, $\{D_t\}$, $\{\epsilon_t\}$, and $\{u_t\}$. That is, a fixed law of motion of the form (20) with given numerical values for the H's cannot be used to investigate the consequence of arbitrarily specified numerical sequences for the W, D, ϵ, and u's. In effect, a particular version of (20) can be expected to hold up only for W, D, ϵ, and u sequences drawn from a restricted domain: namely, sequences obeying the probability laws (8), (4), (5), and (2).[28]

However, until Lucas wrote in 1976, evaluating the effects of interventions in this inappropriate way was the accepted procedure in both the macroeconometric and the microeconometric literatures. Regrettably, to this day it remains the procedure used in the overwhelming majority of analyses of policy interventions. It should be emphasized once again that from the viewpoint of the dynamic decision theory described above, the question of how agents will respond to "arbitrary sequences" of "forcing variables" W, u, ϵ, and D is not well posed. In effect, unless the researcher specifies precisely the perceived laws of motion for the "forcing variables," he has not specified the constraints subject to which decision makers are thought to be acting.[29]

Thus, in order to be able to evaluate interventions operating on the δ's, it is necessary to formulate and estimate the model in terms of the parameters of preferences (A_0, A_1, A_2), technology (f and d), and the constraints (the δ's). The argument in favor of formulating and estimating the dynamic model at the level of the deep parameters $\{A_0, A_1, A_2, d, f, \delta_w, \delta_u, \delta_\epsilon, \delta_D\}$ is in much the same spirit as the usual justification for estimating "structural" parameters rather than reduced-form parameters. As Marschak (1953) argued, the researcher wants to estimate those objects which will permit him to analyze an interesting class of changes in the environment. Dynamic economic theory has forced us to reexamine whether objects long thought to be "structural," including the parameters of decision rules such as consumption, investment, and portfolio balance schedules, are correctly taken to be invariant with respect to changes in the environment. Once agents' behavior is modeled in terms of genuinely dynamic optimization problems, it becomes apparent that the parameters of observed decision rules should not be viewed as structural (see

[28] This message is at least implicit in the work by Lucas and Prescott (1971). Gordon and Hynes (1970) made the argument in an informal way. Lucas (1976) forcefully brought the message to the attention of macroeconomists.

[29] However, some economists continue to argue that existing macroeconometric models can be used to predict the effects of such arbitrary sequences (see Friedman 1978).

Muth 1961; Lucas and Prescott 1971; Merton 1971; and Lucas 1972*a*).

The Neglect of Learning

At this point it is worthwhile to discuss a modification of the preceding kind of setup which several economists have apparently had in mind.[30] For this purpose it is sufficient to consider the problem of maximizing the social welfare criterion subject to the given laws of motion (8), (4), (5), and (2) for W_t, u_t, D_t, and ϵ_t. By relabeling and reinterpreting the variables, we can think of this as a choice problem faced by a single private agent. In posing this problem, it was assumed that the agent solving the problem knows the true values of the parameters of the objective function (17) and the true values of the polynomials in the lag operator $\delta_\epsilon(L)$, $\delta_u(L)$, $\delta_D(L)$, and $\delta_w(L)$. The observation has been made that this setup fails to incorporate a model of how the agent optimally learns about the δ's from observations on past realizations of the forcing variables ϵ, u, D, and W. Presumably, if the agent has only finite histories of observations on ϵ, u, D, and W at his disposal, then at each point in time he is uncertain about the parameters of the polynomials δ. Why not modify the preceding setup to include uncertainty about the δ's and a model of optimal learning about the δ's? There seem to be three reasons why such extensions have not as yet successfully been incorporated into rational expectations models.

The first is as follows. A general model of optimal learning about the δ's is readily available in the "Kalman filter," which can be used to model how a rational agent would use observations on $(\epsilon_t, u_t, D_t, W_t)$ to revise his prior beliefs about the δ's.[31] However, with the δ's uncertain, it is no longer possible to give closed-form formulas for the optimal decision rule in terms of what are now the posterior probability distributions over the δ's. The reason that no one has yet obtained or is likely ever to obtain such closed formulas is as follows. In deriving the closed form of the restrictions (21) for the case in which the δ's are

[30] See Friedman 1979 and Modigliani 1977.

[31] See Anderson and Moore 1979. The Kalman filter provides a model of Bayesian learning about the δ's where the initial prior and the posteriors are multivariate normal. However, as Hansen points out to me, normal posteriors for the δ's are inadmissible for dynamic models of the class described here. This is because the dynamic optimization problems we consider may be ill posed for points in the parameter space of δ's for which the zeroes of det $\delta(z)$ are less than $\sqrt{\beta}$ in modulus. Only priors and posteriors that assign zero probability to this region of the parameter space are in general admissible for our problems. This rules out multivariate normal distributions. Taking account of this admissibility constraint severely complicates the task of building a model of optimal learning about the δ's.

assumed known with certainty, the Wiener-Kolmogorov prediction formula,

$$E_t W_{t+i} = \left[\frac{\delta_w(L)^{-1}}{L^i}\right]_+ \delta_w(L) W_t,$$

was used extensively. Here $[\Sigma_{j=-\infty}^{\infty} \alpha_j L^j]_+ = \Sigma_{j=0}^{\infty} \alpha_j L^j$, so that $[\quad]_+$ means "ignore negative powers of L." The Wiener-Kolmogorov formula is equivalent with the "chain rule" of forecasting (see Shiller [1972] or Sargent [1979] for expositions). These equivalent forecasting rules are known to be correct for the case in which the δ's are known with certainty. However, as Chow (1973) has pointed out, where there is a nontrivial posterior density over the δ's, there is in general no known closed-form formula such as the above one for the i-step-ahead forecast. For example, it is not true that where δ is uncertain, the correct expression for $E_t W_{t+i}$ is given by replacing the δ_{wj}'s with their posterior means in the above formula. The fact that there is no closed-form prediction formula for sufficiently general cases implies that it is impossible to derive closed-form versions of decision rules (and hence equilibria) that correspond to (21). As we shall see, for the kind of empirical work we are advocating, it is important to have a closed form for the mapping from the parameters of the objective functions (17) and the dynamic constraints to the decision rule (20). From this viewpoint, the suggestion that one ought to build a learning mechanism into rational expectations models is not useful in suggesting practical econometric alternatives to the procedures recommended here.[32]

Another drawback with incorporating learning is that, even if one could derive the decision rules in the face of uncertain δ's, the issue would arise of how to determine the prior used to initiate the learning model for the δ's. Would it be imposed a priori or estimated? If the initial prior were to be estimated, this would substantially complicate the estimation problem and add to the number of parameters.

Finally, in many settings the Bayesian learning model implies that the posterior distributions collapse about the true δ's as time passes without limit. In such settings, even if the researcher erroneously assumes that the δ's are known with certainty when in reality agents are learning about them in an optimal way, the researcher continues to obtain consistent estimators of the underlying parameters $\{A_0, A_1,$

[32] Further, notice that if the decision rules could be calculated in closed form under uncertainty about the δ's, the resulting time-series models would have time-varying coefficients and so be nonstationary. Even if calculating the decision rules were a tractable task under uncertainty about the δ's, the loss of stationarity that it would imply might well be a price that the applied economist would not be prepared to pay even in exchange for the "greater realism" of the learning assumption.

A_2, f, d, δ_w, δ_D, δ_ϵ, $\delta_u\}$ using the methods described here and in Hansen (1979) and Hansen and Sargent (1980*a*). It does seem likely that by erroneously ignoring the phenomenon of learning about the δ's, the researcher is incorrectly calculating the asymptotic covariance matrix of his estimators. However, at present nothing is known about the nature of this error. Further, since we simply do not know how to compute optimum decision rules under the assumption that agents know the δ's with uncertainty, no consistent estimators of the underlying parameters have been proposed that incorporate agents' learning about the δ's in the optimal way, to say nothing of expressions for the associated asymptotic covariance matrices.

From the preceding considerations, I draw the conclusion that incorporating optimal Bayesian learning about the δ's on the part of agents is not a research avenue that soon promises appreciable dividends for the economist interested in applying dynamic competitive models of the sort described here.

A Model of the "Error Term"

We now derive a "dynamic supply curve" for the industry by using the industry-wide production function $Y_t = fK_t + \epsilon_t$ to eliminate K from (20) in favor of Y. Multiplying both sides of (20) by f and then adding ϵ_{t+1} to both sides gives

$$\begin{aligned} Y_{t+1} = {} & H_0 f + fH_w(L)W_t + fH_D(L)D_t \\ & + fH_\epsilon(L)\epsilon_t + fH_u(L)u_t \\ & + H_1 Y_t + \epsilon_{t+1} - H_1\epsilon_t. \end{aligned}$$

Eliminating u_t by using $u_t = P_t - A_0 + A_1 Y_t - A_2 D_{1t}$ gives

$$\begin{aligned} Y_{t+1} = {} & [H_0 f - fH_u(1)A_0] + fH_u(L)P_t + fH_w(L)W_t \\ & + [fH_D(L) - fH_u(L)A_2\phi_D]D_t \\ & + [H_1 + fH_u(L)A_1]Y_t \\ & + [1 + fH_\epsilon(L)L - H_1 L]\epsilon_{t+1}. \end{aligned} \tag{23}$$

This can be written as

$$\begin{aligned} Y_{t+1} = {} & S_0 + S_p(L)P_t + S_w(L)W_t \\ & + S_D(L)D_t + S_Y(L)Y_t + S_\epsilon(L)\epsilon_{t+1}, \end{aligned} \tag{24}$$

where

$$
\begin{aligned}
S_0 &= H_0 f - fH_u(1)A_0, \\
S_p(L) &= fH_u(L), \\
S_w(L) &= fH_w(L), \\
S_D(L) &= [fH_D(L) - fH_u(L)A_2\phi_D], \\
S_Y(L) &= [H_1 + fH_u(L)A_1], \\
S_\epsilon(L) &= [1 + fH_\epsilon(L)L - H_1 L].
\end{aligned} \tag{25}
$$

Recall that the demand curve is

$$P_t = A_0 - A_1 Y_t + A_2 D_{1t} + u_t. \tag{3}$$

Using $\delta_\epsilon(L)\epsilon_t = V_t^\epsilon$ and $\delta_u(L)u_t = V_t^u$, we can write the supply and demand curves as

$$
\begin{aligned}
Y_{t+1} = S_0 &+ S_p(L)P_t + S_w(L)W_t + S_D(L)D_t \\
&+ S_Y(L)Y_t + S_\epsilon(L)\delta_\epsilon(L)^{-1}V_{t+1}^\epsilon,
\end{aligned} \tag{26}
$$

$$P_t = A_0 - A_1 Y_t + A_2 D_{1t} + \delta_u(L)^{-1}V_t^u. \tag{27}$$

To discuss identification and estimation of the model, we need a theory about what is unknown to the econometrician. In constructing the model, we have taken the view that all of the variables on the right-hand side of the supply and demand curves (24) and (3) (or equivalently [26] and [27]) are known to the representative firm. Thus, from the viewpoint of private agents, (26) and (27) describe exact linear functions of the right side variables in which there are no "random errors."[33] The only tractable way that has so far been discovered of introducing random errors into (26) and (27) has been to assume that the econometrician has less information than do the private agents. The smaller information set of the econometrician leads to what from his point of view are random terms in relationships to be derived from (24) and (3) or (26) and (27). The idea is to restrict the econometrician's information set relative to that of private agents in a way both that is plausible and that leads to a tractable statistical model of the error term. I shall describe two models of the error term that can be constructed in this way.

One model results from assuming that the econometrician has time series on $\{P_t, W_t, D_t, Y_t\}$ but never observes the random processes ϵ_t and u_t. On this interpretation, ϵ_t and u_t become random terms in (24)

[33] This is a consequence of the fact that the representative firm views itself as playing a dynamic "game against nature," and so finds it optimal to use a nonrandom strategy, that is, a strategy that can be expressed as an exact function of its information variables and other state variables.

and (3) from the econometrician's viewpoint.[34] In constructing the model, we have already imposed that V_{t+1}^{ϵ} is orthogonal to all of the variables on the right side of (26) and will assume that V_t^u is orthogonal to Y_t. We can also impose that V_t^u is orthogonal to D_{1t}, if we wish,[35] although we might get by with a weaker assumption.

The second model of the error term results from assuming that the econometrician sees less of D_t and W_t than do private agents. It is convenient to postpone a detailed discussion of this second model of the error and, instead, first to discuss identification and estimation under the first model of the error term.

Identification and Estimation

With this model of the error terms, we can proceed to discuss identification and estimation. First, notice that every variable that appears on the right side of the demand schedule (27) also appears on the right side of the supply schedule (26). The dynamic economic theory leading to (26) makes the reason for this clear, since any variables that help predict future prices P will appear in the supply schedule of the representative firm. This immediately implies that any variables that help predict the demand shifters D_{1t} will appear in the supply schedule.[36] The fact that no variables on the right side of the demand curve (27) are excluded from the supply schedule (26) means that, if the supply schedule is to be identified, the source of identification must be found in restrictions of a kind different from the usual exclusion restrictions treated extensively in econometrics textbooks.[37] According to the standard "order condition" for identification, equa-

[34] This is a version of the model of the error term analyzed by Hansen and Sargent (1980*a*) and Sargent (1978).

[35] We have assumed that V_t^{ϵ} and V_t^u are the "innovations" or one-step-ahead errors in predicting ϵ_t and u_t on the basis of observations on all variables in agents' information set at time $t-1$ (see n. 6). This implies that V_{t+1}^{ϵ} is orthogonal to all variables on the right side of (26). If we assume that V_t^u is orthogonal to V_t^{ϵ}, it also implies that V_t^u is orthogonal to Y_t. Imposing that V_t^u is orthogonal to D_{1t} amounts to assuming that D_{1t} is strictly exogenous in (27), which is stronger than the Granger causality assumptions already imposed on D_{1t}, namely, that except for lagged D's, no other variables in the model Granger cause D_{1t}.

[36] There is a singular class of exceptions to this statement. In the special case that

$$ED_{1t+j} \mid \{D_{t-s}\}_{s=0}^{\infty} = ED_{1t+j} \mid \{D_{2t-s}\}_{s=0}^{\infty} \quad (28)$$

for all $j \geq 1$, D_{1t}'s will appear in the demand schedule but not in the supply schedule. The condition (28) is usually thought to be exceedingly unlikely for any economic time series $\{D_{1t}\}$.

[37] These remarks about identification should be compared with Milton Friedman's discussion (1953) of the conditions needed for "supply" and "demand" to provide a useful categorization of the factors impinging on price and output. Friedman argued that the categorization was useful to the extent that it effectively sorted forces acting on price and output into mutually exclusive categories.

tion (26) is hopelessly underidentified.[38] Thus if the parameters of the model are to be identified, sources of prior information not of the exclusion variety must be available. The main source of these restrictions in the present model is the extensive body of cross-equation restrictions embodied in equations (21) and (25). Equations (21) and (25) give the parameters of the supply schedule (26) as nonlinear functions of the parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w(L), \delta_D(L), \delta_u(L)$, and $\delta_\epsilon(L)\}$. In general, provided that the parameters r_D and r_w, which determine the order of the autoregressive processes for D and W, and the parameters p and q, the number of elements in the vectors D and W, respectively, are large enough, these cross-equation restrictions identify or overidentify the parameters of the model. The strength of overidentification generally increases with increases in the orders r_D and r_w and the dimensions p and q.[39]

At this point it is useful to collect together the equations comprising the model as

$$\begin{aligned} Y_{t+1} = S_0 + S_p(L)P_t + S_w(L)W_t + S_D(L)D_t \\ + S_Y(L)Y_t + S_\epsilon(L)\delta_\epsilon(L)^{-1}V_{t+1}^\epsilon, \end{aligned} \qquad (26)$$

$$P_t = A_0 - A_1Y_t + A_2D_{1t} + \delta_u(L)^{-1}V_t^u, \qquad (27)$$

$$\delta_w(L)W_t = V_t^w, \qquad (8)$$

$$\delta_D(L)D_t = V_t^D, \qquad (5)$$

where

$$\begin{aligned} S_0 &= +f^2\lambda n\beta A_0 d^{-1} - fH_u(1)A_0, \\ S_p(L) &= +f^2 n\lambda\beta d^{-1}\left\{\frac{L^{-1}[I - \delta_u(\lambda\beta)^{-1}\delta_u(L)]}{1-\lambda\beta L^{-1}}\right\}, \\ S_w(L) &= \frac{-n\lambda\beta f}{d}\phi_w\left\{\frac{L^{-1}[I - \delta_w(\lambda\beta)^{-1}\delta_w(L)]}{1-\lambda\beta L^{-1}}\right\}, \\ S_D(L) &= +f^2 n\lambda\beta d^{-1}A_2\phi_D\left\{\frac{L^{-1}[I - \delta_D(\lambda\beta)^{-1}\delta_D(L)]}{1-\lambda\beta L^{-1}}\right\} \\ &\quad - f^2 d^{-1} n\lambda\beta\left\{\frac{L^{-1}[I - \delta_u(\lambda\beta)^{-1}\delta_u(L)]}{1-\lambda\beta L^{-1}}\right\}A_2\phi_D, \end{aligned} \qquad (29)$$

[38] The fact that the demand curve excludes some variables that appear in the supply schedule is due to the static specification for the demand curve. This feature of the model would not survive a variety of alterations that might plausibly be used to introduce dynamics into the demand curve. E.g., if the demand schedule involved expected future prices as arguments, all variables that help to predict future prices would appear in the econometrically operational expression for current P_t that would correspond to (27).

[39] This characteristic of identification in rational expectations models has been noted in various contexts by several authors, including Lucas (1975) and Sims (1980).

$$S_Y(L) = +f^2 d^{-1} n\lambda\beta A_1\left\{\frac{L^{-1}[I - \delta_u(\lambda\beta)^{-1}\delta_u(L)]}{1 - \lambda\beta L^{-1}}\right\} + \lambda,$$

$$S_\epsilon(L) = 1 - \frac{A_1 f^2 n\lambda\beta}{d}\left[\frac{I - \delta_\epsilon(\lambda\beta)^{-1}\delta_\epsilon(L)}{1 - \lambda\beta L^{-1}}\right] - \lambda L.$$

$$[1 - (\lambda\beta)^{-1}L](1 - \lambda L) = [1 - (1 + \beta^{-1} + A_1 f^2 n d^{-1})L + \beta^{-1}L^2].$$

Equations (26), (27), (8), and (5) form a statistical model for the joint process (P_t, Y_t, W_t, D_t). The model is linear in the variables but is characterized by the extensive set of cross-equation restrictions described by (29). With the model of the error terms currently under discussion, the statistical model of the (P_t, Y_t, W_t, D_t) process has been spelled out sufficiently completely that we could write down the likelihood function for a sample (P_t, Y_t, W_t, D_t), $t = 1, \ldots, T$, assuming a normal probability density for $(V_t^w, V_t^u, V_t^D, V_t^\epsilon)$.[40] Maximum likelihood estimates of the free parameters of the model $\{A_0, A_1, A_2, f, d, \delta_D(L), \delta_w(L), \delta_u(L), \delta_\epsilon(L)\}$ could then be obtained. Computational details of such procedures are described by Sargent (1977, 1978) and Hansen and Sargent (1980*a*). From the point of view of computing the estimates, it is a great practical advantage that (29) gives a set of closed-form formulas for the cross-equation restrictions imposed by the dynamic economic theory.

Application of Bayesian Methods

The fact that for the present model of the error terms it is possible to write down a normal likelihood function means that in principle Bayesian methods are applicable. Letting θ be the list of parameters of the model and Z be the data, we have

$$f_{\text{post}}\{\theta \mid Z\} = \frac{l\{Z \mid \theta\} f_{\text{prior}}\{\theta\}}{f(Z)},$$

or

$$f_{\text{post}}\{\theta \mid Z\} = l(Z \mid \theta) f_{\text{prior}}(\theta) / \int l(Z \mid \theta) f_{\text{prior}}(\theta) d\theta, \tag{30}$$

where $f_{\text{post}}\{\theta \mid Z\}$ denotes the posterior probability density, $f(Z)$ the probability density of Z, $f_{\text{prior}}\{\theta\}$ the prior density on θ, and $l\{Z \mid \theta\}$ the likelihood function. Measures of the location and dispersion of the posterior distribution of θ can be calculated, for example, by integrating $\theta^k \cdot f_{\text{post}}\{\theta \mid Z\}$ over θ for appropriate values of k. In the Bayesian view, the role of data analysis is to trace out in as revealing a way as possible the mapping defined by (30) from the prior to the

[40] See Hansen and Sargent (1980*a*) for a discussion of the details.

posterior distribution. For such an analysis to be practical, it substantially eases matters if the mapping (30) can be characterized analytically so that, for example, posterior moments such as $\int \theta f_{\text{post}}(\theta \mid Z) d\theta$ can be calculated without the need to resort to numerical integration. Zellner (1971) and Leamer (1978) describe forms of prior densities $f_{\text{prior}}(\theta)$ that have the property that the mapping (30) is one that can be written as an analytic closed form when $l(Z \mid \theta)$ is the normal likelihood function.

In the context of dynamic economic models of the class represented by (26), (27), (8), and (5), the question of whether the mapping (30) can be characterized analytically hinges on which parameters one regards as being in the list θ about which the researcher has formulated prior information. One possibility is that θ consists of the S's of (26), the A's of (27), and δ_w, δ_u, and δ_D of (27), (8), and (5). With this interpretation of θ, then since (26), (27), (8), and (5) are linear in the S's, A's, δ_w, and δ_D, it is possible to get analytic characterizations of the mapping from $f_{\text{prior}}(\theta)$ to $f_{\text{post}}(\theta \mid Z)$. For example, Leamer (1972) and Shiller (1972) have shown how priors of various forms on the S's in (26) can tractably be mapped into posteriors, in contexts where (26) is appropriately viewed as a regression equation. In effect, Leamer (1972) and Shiller (1973) provided formal Bayesian methods for imposing restrictions on lag distributions of a general kind, examples of which had long been imposed by applied econometricians. These restrictions usually corresponded to restrictions directly on our S_j's. Predating the work of Shiller and Leamer were the restrictions on distributed lags proposed by Koyck (1954), Cagan (1956), Milton Friedman (1957), Almon (1965), and Jorgenson (1966). There was also the frequently used identifying restriction that various distributed lag weights sum to unity.[41] All of these approaches view the S's themselves as among the free parameters of the model about which the researcher can reasonably be imagined to have formed views summarized by a prior distribution.

Unfortunately, the tractability of the Leamer-Shiller approach is purchased at the cost of ignoring the essential aspects of the dynamic economic theory leading to (26). According to that theory, the S's are not free parameters but are complicated functions of the parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w(L), \delta_D(L), \delta_u(L), \delta_\epsilon(L)\}$. It is this list of parameters about which it seems most appropriate to expect an economist to have prior beliefs. The parameters $\{A_0, A_1, A_2, f, d\}$ are the parameters describing preferences and the technology, about which the economic theorist may have some prior beliefs. The

[41] This restriction was criticized by Lucas (1972*b*) and Sargent (1971) for essentially the same reasons given here.

economists' prior beliefs about the parameters $\{\delta_w, \delta_D, \delta_u, \delta_\epsilon\}$ are presumably on a different theoretical footing than his beliefs about $\{A_0, A_1, A_2, f, d\}$, since the former list simply characterizes the serial correlation properties of the "shift variables" about which economic theory itself suggests little, although casual general observations may suggest a presumption in favor of high serial correlation, at least in some types of variables. In any event, it is the deep parameters $\{A_0, A_1, A_2, f, d, \beta, n, \delta_w, \delta_D, \delta_u, \delta_\epsilon\}$ that must be estimated, if one is to build a model that potentially overcomes Lucas's critique of econometric policy evaluation procedures.[42]

When this list of deep parameters contains the objects of interest, Bayesian analysis using (30) becomes much less tractable. This is because the likelihood function $l(Z \mid \theta)$ becomes a very complicated function of the free parameters in θ, by virtue of the complicated nature of the cross-equation restrictions illustrated in (29). Although Bayesian analysis is still possible, the researcher will be forced to use numerical methods to characterize the mapping from the prior to the posterior given in (30). For example, for a given prior, numerical integration will have to be used to calculate the moments of the posterior distribution. My own judgment is that given current computer technology, formal Bayesian estimation procedures seem prohibitively expensive for most members of the class of dynamic models considered here. This is obviously not an objection to Bayesian methods in principle. However, I believe that the high cost attached to applying Bayesian methods correctly helps to explain why they have not yet been applied extensively to estimating rational expectations models.

A Second Model of the "Error Term"

More serious limitations on the domain of Bayesian techniques emerge if the researcher embraces a second model of the error term, which we now discuss. In the second model of the error term, it is assumed that the econometrician possesses only observations on subsets $\tilde{W}_t \subset W_t$ and $\tilde{D}_t \subset D_t$ of the information variables that private agents use to forecast future w_t's and D_{1t}'s.[43] It is assumed that these

[42] A point related to that raised in n. 29 is relevant here. Priors and posteriors that assign positive probability to points in regions for which zeroes of det $\delta(z)$ are less than $\sqrt{\beta}$ in modulus are inadmissible. This is because under such distributions, for some regions in the parameter space with positive probability, the dynamic optimum problems are not well posed. Taking this into account would substantially complicate the analysis since it would involve using mathematically less tractable distributions.

[43] This model of the error term was originally proposed by Shiller (1972) in a related but somewhat different context. The model was applied in the present context by Hansen and Sargent (1980*a*). Nerlove, Grether, and Carvalho (1979) also recommend Shiller's model of the error term.

subsets of information variables follow autoregressive processes $\tilde{\delta}_D(L)\tilde{D}_t = \tilde{V}_t^D$ and $\tilde{\delta}_w(L)\tilde{W}_t = \tilde{V}_t^w$, where $\tilde{\delta}_D(L)$ and $\tilde{\delta}_w(L)$ are polynomials in the lag operator of order $\tilde{r}_D$ and $\tilde{r}_w$, respectively. Then it turns out that the equilibrium law of motion for capital, (20), can be written in a form identical to (20), except that $W, D, \delta_D, \delta_w, H_D$, and H_w are to be replaced by the corresponding objects with tildes above them, *and* that there appears an additional random disturbance η_t on the right side of (20). The cross-equation restrictions (21) continue to characterize the objects with tildes over them.[44] The random variable η_t can be shown to be orthogonal to all of the current and lagged values of $\tilde{W}$ and $\tilde{D}$.[45] However, it turns out that η_t is in general serially correlated, with serial correlation properties that depend on the joint covariance properties of those variables in D_t and W_t that the econometrician does not have observations on. In the context of this setup, it is not even possible to write down the likelihood function without specifying details of the moments of information variables in D and W that are unobservable to the econometrician. It would seem attractive to adopt an estimation procedure that avoids the implicit theorizing about the stochastic properties of the unobserved D's and W's that an estimator using the likelihood function requires.

One such estimation strategy that exploits the orthogonality of η to $\tilde{D}$ and $\tilde{W}$, without requiring all of the added details required to write down a likelihood function, has been developed by Hansen (1979). The "generalized method of moments" estimators of Hansen have the advantage of delivering estimators of the free parameters whose desirable statistical properties do not depend on any arbitrary assumptions about the serial correlation properties of the η_t's.[46] These generalized method of moments estimators were invented precisely to handle situations in which the researcher is substantially more confident of the orthogonality conditions delivered by his theorizing than he is about the serial correlation properties of the error. These methods construct statistically consistent estimators, while avoiding the need to form the likelihood function. However, in acknowledging that he does not have enough information about the disturbances to construct the likelihood function, the researcher loses the ability to employ Bayesian methods, since knowledge of the likelihood function is essential for using Bayes's law as in (30).

[44] See Hansen and Sargent 1980*a*.

[45] Ibid.

[46] Under regularity conditions provided by Hansen (1979), the estimators of the underlying parameters are shown to be consistent and most efficient within a restricted class of estimators. Hansen's discussion of the conditions for consistency, which also has implications for the conditions for consistency of maximum likelihood estimators, is at this date the key reference on issues of statistical consistency in linear rational expectations models.

Interrelated Industries

In the Lucas-Prescott model of a single industry, state variables which help the firm predict the future prices of inputs appear in the representative firm's decision rule. The laws of motion of these input prices have been taken as given from outside the model. In actuality, the prices of these inputs are usually thought to be determined by trades in another market, one source of demand for which stems from the industry being modeled by Lucas and Prescott. If this other market is modeled explicitly, nontrivial modifications also occur in the analysis of the original industry. Thus, consider the example of a corn-hog model in which part of the output of one industry, corn, is an input into the production of the other industry, hogs. If the technology is such that hog producers have an incentive to forecast future corn prices, it follows that state variables that appear in the laws of motion for the total output and price of corn will also appear in the decision rule of the representative hog producer. Also, because the corn producer has an incentive to forecast the price of corn, which depends partly on the demand for corn from hog producers, the state variables that appear in the laws of motion for total hog output and the price of hogs will appear in the optimal decision rule of the representative corn producer. Hence, each industry inherits the state variables of the other. Furthermore, the equilibria in the two industries must be defined jointly, since the laws of motion for endogenous market-wide state variables are simultaneously determined for the two markets. There continues to be a fictitious social planning problem that a rational expectations equilibrium solves, one that is a version of an interrelated factor demand problem in which the planner is jointly optimizing the performance of both industries.[47]

These remarks indicate that the analyst will often face a hard practical decision about which dynamics he takes as given from outside the model. The internal logic of this class of models tends to propel the analyst toward a general equilibrium formulation in which the laws of motion characterizing each distinct industry are determined simultaneously with the laws of motion for all industries with which it buys and sells. In any given application, the researcher will have to choose what laws of motion he takes as given from outside the model, for the purposes of the analysis at hand.

Conclusions

Remaking dynamic econometric practice so that it is consistent with the principle that agents' constraints influence their behavior is a task

[47] These claims are proved for a particular version of a "corn-hog model" in some unpublished notes (Sargent 1980*a*).

that is far from finished. Further, properly allowing for the implications of the principle will surely require abandoning many presently received ways of interpreting data. A variety of setups can be imagined that are consistent with the principle. For example, a variety of variations of the setup of this paper can be imagined in which agents optimize but have smaller information sets than have been attributed to them here. Also, information discrepancies across classes of agents can be assumed. In many such cases, endogenous variables such as prices will play an important role in conveying information to agents. In models with dynamics as complicated as those of our examples, these variations introduce substantial analytical difficulties. To date there is very little work which investigates the econometric implications of such complications to setups like ours. With or without these complications, building a dynamic econometrics that is consistent with our simple principle from economic theory is a challenging task. It is sure to require substantial changes in the ways that applied economists interpret economic time series.

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