MEASURING MARGINAL UTILITY BY REACTIONS TO RISK

By William Vickrey

Many attempts have been made to measure marginal utility. Some of the methods used have been plausible, others less so, but none of them have been very successful. To some extent the difficulty has been with the inadequacy of the data, but the basic difficulty with most of the actual attempts has been with the nature of the fundamental assumptions required by the methods used. Indeed, it has been maintained that, aside from purely introspective appraisal, marginal utility cannot be measured at all. It may well be that the attempt to derive a utility function from conventional data on consumer budgets, demand curves, prices, or choices is a wild-goose chase after a function that would have a doubtful meaning even if it could be determined. On the other hand, by considering individual reactions to choices involving risks, a more meaningful result may be obtainable.

I. UTILITY AND CHOICES BETWEEN ASSURED ALTERNATIVES

To clarify the issue, let us review briefly the notion of utility and its relation to indifference maps and consumer choice. Studies of consumer choices between definite and certain alternatives tell us only the rank of the various situations in the preference of the individual or group being studied. This will be true whether the data are sought by asking questions of individuals as to their choice between hypothetical alternatives, or whether the data consist of the observed choices of individuals confronted with actual income and price situations. If it be assumed that these choices are consistent, all of this information can be contained in an indifference map. If the marginal utility can be inferred unequivocally from such studies it can also be inferred from the corresponding indifference map, for the choices themselves can be inferred from the map. Conversely a utility function that cannot be inferred from such a map must be dependent upon some additional data or on some unsubstantiated assumption.

An indifference map can be described by a function $U(x, y, z, \cdots, w)$, each value of U giving an indifference contour passing through all those combinations of quantities (strictly, rates of consumption) of the several commodities x, y, z, $\cdots$, w, which are deemed equally desirable by the individual concerned. Combinations giving larger values for U will be those deemed more desirable, those giving smaller values for U less desirable. But any other function $V = V(U)$ will also give the same indifference map, and will therefore describe the behavior of the individual just as effectively as U, provided only that the function is monotone increasing, that is, that $V'(U)$ is everywhere positive. We can for example put $V=U^2$, $V=\log U$; in terms of the map itself there is in general no meaningful criterion for determining which of the many admissible functions $V(U)$ is *the* utility function or which function V_x'/P_x should be taken to represent the "marginal utility of money."

In some cases, however, we may be able to find a function V that has certain unique properties. If among the various functions that are capable of describing a given indifference map there exists one for which V_x' is independent of the quantities of all commodities other than x, then to be sure this V is unique. For if there is some other function W that describes the given indifference map and has a first derivative W_y' that is independent of the quantities of commodities other than y, the W is necessarily but a linear function of V and can be considered identical with V. To show this we put $V'(X)=f(x)$, and upon integrating get $V=F(x)+G(y, z, \cdots, w)$. If then we have some other function $W(V)$ for which $W_y'=h(y)$, then $h(y)=W_y'=W'(V)V_y'=W'(V)G_y'$ and $W'(V)=G_y'(y, z, \cdots, w)/h(y)$; thus $W'(V)$ is independent of x. But since V depends on x, $W'(V)$ must not depend on V; i.e., $W'(V)$ must be a constant and W a linear function of V. V and its linear transformations are therefore the only functions that possess such a property.

Most attempts to determine marginal utility hitherto have been based on the assumptions first that some utility function can be found that will thus make the marginal utility of some commodity independent of the quantities of all other commodities and second that this function when found has some special validity for the purposes at hand. The methods suggested by Irving Fisher,[1] and the isoquant, quantity-variation, and translation methods of Ragnar Frisch[2] all involve such assumptions. If the assumptions were well founded and the data adequate, all of these methods would tend to produce the same result. They differ chiefly in the data used, in the population covered, and in the additional assumptions made as a partial substitute for more complete data.

On the other hand to say that the marginal utility of one commodity x is independent only of the quantity of some other commodity y but not necessarily of all the remaining commodities $z, \cdots, w$ is insufficient to establish a unique utility function, for there exist utility functions that can be transformed by a nonlinear transformation in such a

[1] Irving Fisher, "A Statistical Method for Measuring Marginal Utility," in *Economic Essays in Honor of J. B. Clark*, New York, 1927, pp. 157–193.

[2] Ragnar Frisch, *New Methods of Measuring Marginal Utility*, Beiträge zur Oekonomische Theorie, Tübingen, 1932, 142 pp.

way that this criterion is satisfied with respect to other commodities. For example if we consider the function $U=(x^a+y^a)(z^a+w^a)^b$, then on this utility scale $U_x'=ax^{a-1}(z^a+w^a)^b$ is independent of y but not of z or w, and similarly the marginal utility of y is independent of x. If now we consider another utility scale $V=\log U=\log (x^a+y^a)+b \log (z^a+w^a)$, then on this scale $V_x'=ax^{a-1}/(x^a+y^a)$, so that the marginal utility of x is now independent of z and w but not of y, and similarly for z and w. Or we could put $W^b=U$, and on this third scale the marginal utility of z, W_z', will be independent of w, and that of w independent of z, while both will depend on x and y. Thus if all that is stipulated is that the marginal utility of some one commodity shall be independent of the quantity of some other, we could conceivably find as many significantly different utility functions satisfying this condition as there are commodities in excess of one.

It is more than likely that the widely varying results obtained with different commodity combinations indicate that at most the independence criterion is satisfied only as between pairs of commodities or groups of commodities. If so, the use of different pairs of commodities would result in different utility functions even with complete and accurate information. Thus the widely divergent results obtained may not be entirely the result of the inadequacy of the data, but may be inherent in the method used.[3]

Assuming that there exists a utility function for which the marginal utility of some commodity x is independent of the quantity of all the others, we must be able to express the price of x in terms of any other commodity y in the form

$$P_{x/y}=\frac{f(x)}{g(y, z, \cdots, w)}$$

and similarly for $P_{x/z}, \cdots, P_{x/w}$. However, it is sufficient to show for some one other commodity, say y, that

$$\frac{P_{x/y}(x, y, z, \cdots, w)}{P_{x/y}(x, y_0, z_0, \cdots, w_0)}=\frac{P_{x/y}(x_0, y, z, \cdots, w)}{P_{x/y}(x_0, y_0, z_0, \cdots, w_0)}$$

for some initial sets of values $x_0, y_0, z_0, \cdots, w_0$, and for all other values of $x, y, z, \cdots, w$. It may well be doubted whether it will ever be possible to find a commodity x which will satisfy these rather rigorous conditions. Even if an "equivalent commodity" made up of some group or combination of commodities is used, together with some price index

[3] Cf. the results obtained by James N. Morgan in "Can We Measure Marginal Utility," Econometrica, Vol. 13, April, 1945, pp. 129–152, especially the chart on p. 133.

for the group, the likelihood of finding these conditions fulfilled over the entire range of incomes and over substantially the whole range of consumption patterns is remote.

The possibility of finding such a "commodity" is somewhat improved if we are content to find these conditions satisfied only over a given range of utility. If we can find one commodity or combination whose marginal utility can be made independent of quantities of other commodities by a suitable choice of utility function between the utility contours A and B, and another "commodity" for which the marginal utility can be made independent between B and C by using some other utility function, we might proceed to splice together several utility functions and thus cover the whole range of utility.

But even assuming that we could in this way find a utility function that would be unique in satisfying this independence criterion, there is still grave doubt as to what significance is to be attached to it. For example, we might well come across an indifference map describable by the function $U=ye^x$; obviously this U is not an acceptable measure of utility, for U_{xx}'' is positive. There is however a transformation $V=\log U=x+\log y$ which satisfies the independence criterion, but here too V is hardly acceptable as a measure of utility for V_x' is a constant, whereas we normally expect a declining marginal utility. $W=\log V$ or $Z=V^{1/2}$ would be more acceptable as measures of satisfaction, but neither of them satisfy an independence criterion.

Of course, if the utility function is to be used only in ways that are related to this independence property or its corollaries, there can be no objection. But constructors of such utility functions usually do not stop there: they almost always infer that this function (or at least the function that they are sure they could get if only the data were more tractable) rather than any other is the most suitable measure of the satisfactions represented by the different contours of the indifference map and hence of the "sacrifice" involved in descending from one contour to a lower one. Usually in the background there is the objective of translating the "equal sacrifice" or "proportional sacrifice" criteria into actual schedules for income-tax progression. This step, however, cannot be inferred from the independence property alone, but rests on a separate and unsupported assumption.

There are in other words two separate fundamental assumptions involved in measuring marginal utility by any of the independence methods: The first is the factual assumption that the indifference map of the individual or group being studied is such that it can be described by a function of the form $F(x, y, z, \cdots, w)=X(x)+G(y, z, \cdots, w)$. This assumption could be proved or disproved in any given case if sufficient data were available. The second assumption is that this particular function rather than any other such as F^2 or log F is the proper measure of utility or sacrifice. This assumption neither can be verified in terms of the usual data on consumer choice, nor does it appear to be supported by any compelling a priori reasoning. An "independence utility function" may exist and may even be discovered by statistical investigations without necessarily bearing any relation to subjective pleasure or utility. As a method of predicting consumer behavior in response to price changes, one utility function has no advantage over any other, as long as we are dealing with a riskless static equilibrium. As a basis for determining the proper progression of the tax structure, an independence utility function has no demonstrable competence. The jump from certain rather obscure properties of the behavior of individuals in response to price changes to conclusions as to the desirable distribution of income is a rather drastic one, and requires more justification than a mere casual identification of a function that satisfies an independence criterion with one that is to be summed over all individuals and maximized.

Other approaches to the problem of measuring marginal utility have been suggested, but no attempt has as yet been made to translate them into numerical results. In general such methods rely on data not contained in the ordinary indifference map.

Pareto has suggested that the rapidity with which consumers adjust to new conditions may be an indication of the absolute magnitude of the change in their utility that is brought about by their adaptation to the change.[4] But it is doubtful whether any uniquely valid scheme for relating such rates of change to utility can be set up, and in any case the statistical problem of discovering the quantitative relation, assuming that it exists, would be almost insurmountable. Pareto also supposes that in addition to finding an independence utility function in the case where choices are consistent, utility might be defined uniquely in the cases where the integrability conditions are not satisfied and "utility" depends not only on the consumption pattern achieved, but on the path by which it was reached.[5] As it is difficult either to determine such a function or to define its significance for social policy if it could be determined, this possibility also offers little hope.

Paul A. Samuelson has suggested that a clue to the shape of the utility function may be obtained by observing the choices of individuals

[4] "Pour traiter un problème de dynamique, il faudrait connaître non seulement le sens dans lequel l'individu se meut, mais encore l'intensité du mouvement correspondant à une certaine valeur de l'accroissement de la fonction-indice." Vilfredo Pareto, "Economie Mathematique," in *Encyclopédie des Sciences Pures et Appliquées*, Tome 1, Vol. 4, Fascicule 4, p. 598.

[5] *Ibid.*, p. 614.

concerning the distribution of consumption through time.[6] However, this method rests on assumptions that the utility at one point of time is independent of consumption at other points of time, that the utility function is not subject to change through time, and that some schedule of subjective discounts of future utilities can be postulated a priori. These assumptions are by no means easy to justify, and when in addition one is in practice confronted with consumers who have only a very limited freedom to adjust their consumption through time, who are uncertain about the future, and who face constantly changing consumption possibilities, prospects for such a method are almost nil.

F. Zeuthen has also made a similar suggestion: "Such a choice [between *changes* in utility positions] may appear . . . where the same will control several non-simultaneous consumptions, ..."[7] Apparently he intends to refer primarily to distributions of consumption through time, but the quotation also suggests situations where the head of a family for example controls the consumption of the various members. However, it would be necessary to know on what principle the individual controlled the consumption of the various members; the only workable assumptions are either that the marginal utility of the various members be made equal or that the total utilities of the members are adjusted according to some norm. Even so, the results give us only a comparison of the marginal or total efficiency of the various members of the family as "utility machines" as judged by the head of the family; we get little or no information on the shape of the utility function of any one individual. Indeed, on the usual postulate of equal utility functions for all individuals, an equal distribution of outlays among members of the family would have to result unless the head of the family were to some degree "selfish" and arrogated to himself a larger share in the family resources.

II. UTILITY AND CHOICES BETWEEN RISKY ALTERNATIVES

A more promising approach involves using the reactions of individuals to choices involving risk. As long as an individual is confronted only with situations involving merely the substitution of one commodity for another in ratios governed by prices, the question is merely one of preferences as between one certain situation and another certain situation, and utility is defined only as to rank and not as to magnitude. But if the choices available to individuals involve elements of risk, the behavior of individuals can be made to define uniquely a utility func-

[6] Paul A. Samuelson, "A Note on Measurement of Utility," *Review of Economic Studies*, Vol. 4, February, 1937, pp. 155–161.

[7] F. Zeuthen, "On the Determinateness of the Utility Function," *Review of Economic Studies*, Vol. 4, June, 1937, pp. 236–239, esp. p. 237.

tion that can be interpreted directly in terms of sacrifice. Such an approach has been suggested by von Neumann and Morgenstern[8] and by F. Zeuthen.[9] It promises to give more concrete meaning to utility and marginal-utility functions, and appears to be more directly related to problems of social policy such as the distribution of income.

Consider an individual who is faced with a choice between two alternative courses of action, one of which will land him on utility contour B, while the other course has equal chances of landing him on either contour A or contour C. We may consider that a rational choice between these two alternatives would involve the maximizing of the mathematical expectation of some utility function. That is, we may define rational behavior as a behavior capable of being described by the maximization of the mathematical expectation of some function, and we may define utility as the function whose mathematical expectation is thus maximized. If then we can discover situations in which individuals choose "rationally" among alternatives involving risks, and can observe these reactions in sufficient detail, we should be able to discover the "utility" function so defined, at least to within linear transformations. In the above case if the individual is on the margin of indifference as to whether he will choose the certainty of B or the equal chance of A or C, then the difference between $U(A)$ and $U(B)$ is the same as the difference between $U(B)$ and $U(C)$. Similarly, if an individual is on the margin of indifference between the certainty of B and a situation in which the probability of landing on contour X is k and that of landing on A is $1-k$, then $U(B)=kU(X)+(1-k)U(A)$ or $U(X)=U(A)+[U(B)-U(A)]/k$. The utility function is thus determined except for the arbitrary assignment of values to two initial contours of the map; this is equivalent to the selection of a zero point and a unit of measurement.

Samuelson has raised the objection that the ranking of changes from one indifference curve to another is not of itself sufficient to determine the utility function to within linear transformations.[10] The mere ranking of the differences $D(a, b)$ between the various indifference surfaces does not necessarily imply that D must be a cardinal quantity. However, any system of ordering of changes that does not admit of such cardinal expression involves a type of behavior that can be characterized as either irrational or as containing dynamic elements

[8] John von Neumann and Oskar Morganstern, *Theory of Games and Economic Behavior*, Princeton, 1944, 625 pp., esp. pp. 15–20.

[9] *Loc. cit.* in note 7.

[10] Paul A. Samuelson, "The Numerical Representation of Ordered Classifications and the Concept of Utility," *Review of Economic Studies*, Vol. 6, October, 1938, pp. 65–70.

that are not covered by the utility function of a given person at a given time. Rational individual behavior can be considered to require that the ranking of a given change be unaffected by the number of steps in which that change is made. If this postulate be granted, then if changes can be ranked they can also be measured, and accordingly utility can be measured. For then any change $D(a, c)$ can be subdivided into a number of subchanges $D(a, b_1)$, $D(b_1, b_2)$, $D(b_2, b_3)$, $\cdots$, $D(b_{n-1}, c)$ such that each subchange is of the same rank as the unit change $D(a, b_1)$. The change $D(a, c)$ can then be assigned a magnitude corresponding to the number of such unit changes.

This method of course fails if the desirability of a given change depends on the number of steps. This might be the case, for example where there is a utility or disutility attached to changes per se. But where we are considering not changes representing actual successive situations in time but merely differences between alternative hypothetical states to be enjoyed at the same future point of time, such a utility or disutility of change is irrelevant. But even so, the utility function obtained might depend on the starting point from which the various alternatives are to be reached. To state this possibility in another way, the utility associated with a given situation may depend not only on the situation attained but upon the path and the rate at which it was reached. This is essentially Pareto's case of nonintegrability, and involves again either dynamical considerations or irrational behavior on the part of individuals. For practical purposes, we may consider the additional postulate required for the conversion of a ranking of changes into a cardinal utility function sufficiently well satisfied.

When we look at the possibilities of applying this theoretical analysis to actual data, however, and so deriving an actual utility function, the prospects are not encouraging. In the field of business ventures, it is not possible to ascertain with any great definiteness what the a priori probabilities of various degrees of success or failure are. Indeed for the purpose of this analysis it is not so much the objectively appraised chances of success or failure that are needed but rather the investor's own estimate of these chances. These subjective estimates are still less capable of being observed, although they might possibly be elicited by questionnaire.

Possibly we can come closer to finding the required data in the field of insurance. We can consider most forms of insurance as methods of exchanging risky positions for less risky ones. The mathematical expectation of the money income in the less risky insured position will be less than the mathematical expectation of the money income in the uninsured position by the amount absorbed by the insurance company in overhead and profits. But here again, although we are in a better position to gauge the objective probabilities of loss, it is by no means certain that the average purchaser of insurance has any accurate notion of these probabilities. Indeed, most purchasers of insurance would probably be quite surprised if they were told how large a fraction of their premiums is required to pay overhead expenses and profits as compared with the fraction returned to policyholders as dividends and indemnities.

Even assuming that this subjective bias can be dealt with, we still have the problem of deciding over how long a period a given loss is to be spread. For example, if a $10,000 house burns down, are we to consider this as equivalent to decreasing the owner's income from $6,000 to $1,000 over a period of two years, or as equivalent to a decrease from $6,000 to $4,000 over a period of five years? Possibly some answer could be obtained to this question from a study of the relations between fluctuations in income and fluctuations in the expenditures of individuals; as yet, however, the data needed for such a study are not available in suitable form.

Assuming that policyholders are informed and rational, and that we can form some judgment of the period over which uninsured losses should be spread, we might be in a position to derive some conclusions about a utility function from a study of the degree to which persons at different economic levels carry the practice of insurance against casualties, as compared with the net overhead cost of such insurance. One would expect to find that the wealthier classes would tend to insure only relatively large risks, since small casualties could produce only a small differential in the marginal utility. At the other end of the scale, the poorer classes would have more occasion for insurance, since their economic position is relatively insecure. But since insurance in small amounts is comparatively costly, such persons might still be found to insure only against the more important calamities. Also since insurance depends to a large extent on sales effort, and information about the various types of insurance available is not too widespread, the lower income classes may fail to take full advantage of insurance, particularly as agents tend to concentrate their sales efforts on prospects promising larger commissions. Moreover, in such comparisons account would have to be taken of the fact that the income tax discriminates in favor of life insurance and against most forms of casualty insurance.[11] But as the degree to which this is generally appreciated by taxpayers is uncertain, appraisal of this factor is difficult.

Furthermore, there is abundant evidence that individual decisions

[11] William Vickrey, "Insurance under the Federal Income Tax," *Yale Law Journal*, Vol. 52, June, 1943, pp. 554–585, esp. pp. 555–556, 563–564.

in situations involving risk are not always made in ways that are compatible with the assumption that the decisions are made rationally with a view to maximizing the mathematical expectation of a utility function. The purchase of tickets in lotteries, sweepstakes, and "numbers" pools would imply, on such a basis, that the marginal utility of money is an increasing rather than a decreasing function of income. Such a conclusion is obviously unacceptable as a guide to social policy. A small fraction of such gambling can be attributed to the presence of an eleemosynary element. But for the bulk of such gambling the explanation must be sought elsewhere. One explanation that is consistent with maintaining the assumption of rationality in other dealings would be that the purchase of lottery tickets represents the purchase of a right to hope, however forlornly, in a situation otherwise intolerably barren of this psychological necessity. Other forms of gambling can perhaps be ascribed to the persistence of an egoistic delusion that one's own skill or judgment is better than the opponent's, or to utilities derived in the process rather than from the end result.

Even with insurance, considerations other than the maximization of the mathematical expectation of utility enter the picture. Insurance may be taken out because it is the "sound" thing to do, without much thought as to the net cost. Or insurance may be purchased to avoid the responsibility for taking precautions against the casualty involved. Jewelry, for example, is often insured against theft or loss not so much because the loss would seriously affect the economic status of the owner, but rather to avoid the worry which the possibility of loss would otherwise cause. Indeed, it is probably true to say that if the jewelry is so valuable in relation to the income of the owner that insurance would be justified on grounds of maximizing the expectation of utility alone, then the ownership of the jewelry is in itself an unwise extravagance.

In practice, the outlook for actually determining marginal utility by this method is thus not bright. If the determination can be made at all from data on observed behavior, data relating to insurance probably offer the best field for the attempt. But even this may fail, and it may be necessary to resort to the uncertain procedure of asking hypothetical questions, with all the attendant possibilities for misunderstanding and for bias arising from differences between what people think they would do and what they actually would do under the hypothetical circumstances.

III. UTILITY AND THE OPTIMUM DISTRIBUTION OF INCOME

From a theoretical point of view, however, the "risky choice" approach to the measurement of marginal utility does offer the advantage that it ties in rather directly with questions of distribution of income and the proper graduation of progressive taxation, particularly insofar as these questions are conceived of in terms of maximizing the aggregate utility. If utility is defined as that quantity the mathematical expectation of which is maximized by an individual making choices involving risk, then to maximize the aggregate of such utility over the population is equivalent to choosing that distribution of income which such an individual would select were he asked which of various variants of the economy he would like to become a member of, assuming that once he selects a given economy with a given distribution of income he has an equal chance of landing in the shoes of each member of it. Unreal as this hypothetical choice may be, it at least shows that there exists a reasonable conceptual relation between the methods used to determine utility and the uses proposed to be made of it. With the independence method of determining a utility function, there is no obvious connection between the operational definition of the function and the uses to which it is to be put.

Assuming that the marginal utility of money declines with increasing income, maximizing the total utility derived by a population from a given fixed aggregate income implies that this income be distributed equally, due allowance being made for varying needs. On such a basis, the exact shape of the utility function is irrelevant to a determination of the proper distribution. But the aggregate amount of income to be distributed cannot in practice be considered independent of the way in which it is distributed. It is generally considered that if individual incomes were made substantially independent of individual effort, production would suffer and there would be less to divide among the population. Accordingly, some degree of inequality is needed in order to provide the required incentives and stimuli to efficient cooperation of individuals in the production process. As soon as the need for such inequality is admitted, the shape of the utility curve becomes a factor in determining the optimum income distribution.

With these practical effects to consider, the question of the ideal distribution of income, and hence of the proper progression of the tax system, becomes a matter of compromise between equality and incentives. If the total income to be distributed can be taken as a function of the degree of inequality, we can still express the solution in terms of maximizing the aggregate utility. The conditions will naturally be more involved than the mere partition of a fixed total. The appropriate solution would lie somewhere between the extremes of complete equality of income on the one hand and that degree of inequality needed to maximize total income on the other.

What is involved in such a problem may perhaps be made clearer by formulating it in mathematical terms. By means of taxation, (or in some cases a subsidy) we are to establish a relation $r=r(z)$ between the output z_n of the nth individual and his income, r. The output itself will be a function of the individual's income, r, the incentives offered, dr/dz, and the characteristics of the individual himself:

$$z_n = f_n\left(r, \frac{dr}{dz}\right).$$

We wish to choose the function $r(z)$ in such a way as to maximize $\sum U$, where $U=U(r)$ is the utility arising from an income of r. Normally it is necessary to assume that U is the same function for all individuals. We have also to satisfy the condition that the net revenue equal a given required amount: $\sum(z-r)=R$. In form, this is a problem in the calculus of variations.

Unfortunately, in arriving at a solution to this problem, we are even more in the dark about the effect of incentives on individual output and of inequality of income distribution on the total output of the community than we are about the utility function. A wide variety of opinions have been expressed on the subject, ranging from communalists and egalitarians declaring that pride of workmanship and social approval are all that is necessary to elicit the full cooperation of individuals in production, to doctrinaire advocates of laisser faire maintaining that each individual must be allowed to retain as nearly as possible the full marginal product of his labor if maximum output is to be maintained. The prospects for resolving these conflicting opinions and obtaining a convincing quantitative estimate of the relation between the distribution of income and the national output are probably not much better than that for the determination of the utility function itself. The reactions of individuals to different scales of remuneration and different degrees of tax progression are so involved with other factors that the best available statistical techniques applied to the most detailed and extensive data will have a hard time isolating the relation between reward and effort.

Some of the difficulty may be side-stepped by reformulating the problem by means of some simplifying assumptions. If the utility function is made to depend not only on income r but also on productive effort w (thus in effect including the utility of leisure in the problem), we may suppose that the effort put forth by the nth individual will be such as to maximize $U(r, w)$. Suppose the work w required of the nth individual to produce an output z is given by $w=w(z, n)$. Again we wish to maximize $\sum U$ by varying $r(z)$ subject to the condition $\sum(z-r)=R$.

If we substitute $w(z, n)$ in $U(r, w)$, we can get a function $V(r, z, n) = U(r, w)$. The nth individual in adjusting his effort for maximum V will put $V_z'+V_r'p=0$, where $p=r'(z)$ and $q=r''(z)$. This defines a relation between z and n; if we put $G(z, n)=V_z'+pV_r'=0$ where the variable r has been eliminated by substituting $r(z)$, we can then put $G_z'dz=-G_n'dn$. Using a Lagrange multiplier, we can take care of the revenue condition by maximizing

$$\int [U+\lambda(z-r)]dn = -\int [V+\lambda(z-r)]\frac{G_z'}{G_n'}dz$$

where for convenience we shift to a continuous integral instead of a sum of discrete elements. If we now put

$$J(z, r, p, q) = [V+\lambda(z-r)]\frac{G_z'}{G_n'},$$

the Euler equation is

$$J_r' - \frac{d}{dz}J_p' + \frac{d^2}{dz^2}J_q' = 0.$$

This reduces to:

$$(V_r'+\lambda)G_z'+G_n'\frac{d}{dz}\left[\frac{\lambda V_r'(1-p)}{G_n'}\right]-\lambda(1-p)[V_{rr}''p+V_{rz}'']=0.$$

Expanding this expression in terms of U, w, r, and their derivatives produces a completely unwieldy expression. Thus even in this simplified form the problem resists any facile solution.

These formulations are in effect a refinement of the "minimum sacrifice" criterion as suggested by Edgeworth[12] and others. This criterion is by no means universally accepted, however. Simons rejects the whole notion of sacrifice as a criterion for tax progression,[13] while others have argued for "proportional sacrifice" or "equal sacrifice." "Proportional sacrifice" is meaningless unless some arbitrary zero is chosen for the utility function. Nor does choosing the point of minimum subsist-

[12] Francis Y. Edgeworth, "The Pure Theory of Taxation, III," *Economic Journal*, Vol. 7, December, 1897, pp. 550–571; also reprinted in Edgeworth, *Papers Relating to Political Economy*, London, 1925, Vol. II, pp. 63–125.

[13] Henry C. Simons, *Personal Income Taxation*, Chicago, 1938, 238 pp., esp. pp. 5–27.

ence solve the problem, as marginal utility may not be integrable to this point.[14] Fisher, without giving the matter much thought, stated as though it were a matter of course that the proper criterion is equal sacrifice, recognizing explicitly that whether this results in regressive or progressive taxation depends on the elasticity of the marginal-utility function.[15] He may of course have merely selected the only criterion according to which the utility function by itself would give an unequivocal and acceptable answer to the progression problem.

Even though the assumption that choices in situations involving risk are based on maximizing the expectation of a utility function may not hold true, a utility function derived by the use of this assumption may still be a reasonable guide to the optimum distribution of income. If individuals prefer to have the spice of adventure that comes from taking a sporting chance once in a while, this behavior will tend to produce a less sharply declining marginal-utility curve. Relating this marginal-utility curve to a given production-distribution relationship will result in a more unequal distribution of income being indicated as the optimum adjustment. If we assume that individual preferences for taking chances individually are indicative of a corresponding preference for a wider range of possible variation in income, this more unequal distribution of income may be considered an allowance for this preference.

We cannot be certain that this indicated shift in the distribution of income is of the proper amount, however. For example, if the speculative propensity were strong enough so that the "marginal utility" curve derived from observed risky choices increases with income over some range, then the indicated distribution of income might become bimodal; such a result is almost certainly overshooting the mark.

Unlike the "risk utility" function, there is no obvious relation between the method used in obtaining the "independence utility" function and its use as a criterion for evaluating distributions of income. In fact there is no obvious reason to suppose that there will be any relation between a "risk utility" function and an "independence utility" function, even if the latter is found to exist. We have then in the risk method of determining utility not merely a new method of arriving at a utility function, but an entirely new function with new properties. These properties may be similar to those erroneously ascribed to the independence utility function, but this of itself does not establish any relationship.

[14] See Robert L. Bishop, "Consumer's Surplus and Cardinal Utility," *Quarterly Journal of Economics*, Vol. 57, May, 1943, pp. 421–449.

[15] Fisher, *op. cit.*, p. 185.

Determining a utility function by reference to choices involving risk, while simple in theory, is not easy in practice. Such a procedure does not entirely avoid the making of assumptions that may be seriously remote from reality. A risk utility function is also likely to be considerably more difficult to determine from the available data than an independence utility function. Yet if and when it is determined, it will be considerably more definite in concept and more logically applicable to problems at hand.

Civilian Public Service Camp No. 114
Bluemont, Virginia