Notes on the new urban economics

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Since about 1970, ten or twenty papers have been written making use of control theory or programming to analyze optimum or market equilibrium urban land use. The purpose of these notes is to survey and evaluate this new approach to urban economics. Most contributions to the new urban economics assume that employment is concentrated at the urban center, but that housing production functions permit the amount of housing per unit of land to vary with economic conditions. An optimum or equilibrium pattern of housing density is deduced as a function of distance from the center. All contributions include assumptions about the urban transportation system, and some have congestion built into the model. The last part of the paper is a discussion of discrete and continuous representation of urban space. The two possibilities lead to different mathematical representations, and we suggest that discrete representations may be more useful for many purposes.

1. Introduction

■ "All theory depends on assumptions which are not quite true. That is what makes it theory."

Robert Solow

Urban model building has a considerable history. Throughout the 1960s urban specialists built models of the spatial organization and growth of urban areas. A good survey is the work of Brown *et al.*[1] These models had various purposes, but mostly they were intended to assist governments in planning the provision of public services and in planning land use regulation. Most contained considerable detail and many nonlinear relationships, and were solved on computers by ad hoc methods. Most were the work of noneconomists and contained assumptions economists find implausible. However, the most recent and most sophisticated model in this tradition was built at the National Bureau of Economic Research and is the work of economists.[2]

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[1] In [3].

[2] See [5].

In the 1970s, a new genre of mathematical urban models, almost unrelated to the earlier tradition, has appeared. The new models are the subject of these notes and constitute the new urban economics in our title. The hallmark of the new urban economics is the use of fairly sophisticated mathematics—calculus of variations, programming and control theory—to characterize some fundamental aspects of urban structure. Incidentally, the new game is played by almost entirely new players. They are mostly general economic theorists who have recently turned their attention to the urban economy. The old and still thriving game is played mostly by people with longstanding professional interest in urban issues.

The new urban economics is a small but flourishing business. There are perhaps a baker's dozen of published papers, most notably the three other papers in this symposium, four papers in the March 1972 issue of the *Swedish Journal of Economics*, and several papers in various recent issues of the *Journal of Economic Theory*. Counting unpublished papers we have read or seen mention of, there are probably close to two dozen contributions to the new urban economics.

2. Characteristics of the new urban economics models

■ Although new urban economics models differ from each other in important aspects, they characterize the city, or urban area, somewhat as follows. The city is built on a flat plain and travel is assumed to be equally costly in all directions. The city has a well-defined and predetermined central business district (CBD), and may have a pie slice of land removed from it to allow for natural features such as harbors or mountains. The CBD is usually assumed to be of fixed size and to employ a fixed number of workers, most typically the city's entire labor force. Most models do not describe the employment sector in further detail, although some recent papers are notable exceptions. For example, Mirrlees and Dixit introduce economies of scale in the production sector, and Solow introduces diffused local employment in addition to that in the CBD.[3]

In virtually all models, the only travel is commuting trips of the labor force between places of residence and work places in the CBD. Travel within the CBD is usually ignored. Thus, the only spatial characteristic of any location in the city that matters is distance from the CBD or, equivalently, from the city center. This is very important because it means that the residential area of the city can be treated as if it were one-dimensional. A one-dimensional representation of location seems to be crucial if the calculus of variations or control theory is to be used. Distance from the city center plays the role of time in more conventional applications of these techniques.

Travel either costs money or reduces utility (but not both so far, apparently for technical reasons). Thus, other things equal, people would like to live as close as possible to the CBD. But travel is of course not the only thing about which people have preferences. In earlier models, such as those of Muth and Mills,[4] it was assumed that everyone's demand for housing could be described by a particular function of income and the price of housing. More recent models, such as those of Solow and Dixit,[5] postulate utility functions defined

[3] In [11], [4], and [19], respectively.

[4] In [13] and [8], respectively.

[5] In [17] and [4], respectively.

on consumer goods and housing. The latter approach is theoretically preferable, although the former has the advantages that the models are easier to work with and that the parameters of the demand for housing function can be verified empirically. There is nothing to prevent land from entering the welfare function directly, as in Mirrlees,[6] but in most models it enters only indirectly via a housing production function which transforms goods and land into housing services.

In most models, the population of the entire urban area is given, and everyone is assumed to have the same utility or demand functions. People are allowed to live anywhere outside the CBD, and since they can travel in any direction at the same cost, the residential section of the city is doughnut-shaped. The size of the city is determined by incomes, tastes, housing technology, the cost and/or speed of travel, and how much must be paid to bid land away from nonurban uses such as agriculture.

The principal characteristics of most theoretical urban models are those described above. Particular models often have additional features, of course. For example, Muth[7] and others have examined what happens when there are two or more groups of people with different incomes but the same utility functions. And a number of recent models, such as those of Dixit, Mills, and Solow,[8] allow the speed or cost of travel to vary with traffic congestion and the amount of land devoted to transportation to vary with distance from the CBD.

So far, all these models have been static. It is too easy to criticize static models, and we do not intend to do so here. The problem is that theoretically satisfactory dynamic models seem to be enormously difficult to formulate and solve. Even for nonspatial systems economists know very little about dynamic general disequilibrium systems; and it is hardly fair to expect urban economics, where the spatial aspect makes everything so much harder, to progress faster than general economic theory. For most problems that are theoretically interesting, the choice seems to be between static models and no models at all. Those who practice the new urban economics have chosen the former.

Urban models can be solved in two quite different ways. Some models are normative, and for them a solution is an allocation of people, goods, housing, and land at each distance from the CBD which maximizes a social welfare function. Other models are positive, and for them a solution is a competitive equilibrium. The normative models are mostly solved by variational methods or control theory, whereas the positive models tend to be solved by ad hoc methods. In both cases explicit solutions are often difficult or impossible to find, and numerical analysis often has to be resorted to.

3. Implications

■ What has the new urban economics taught us? The most common focus has been on equilibrium or optimum population or housing density as a function of distance from the CBD. Remarkably simple models produce population densities that decline with distance from

[6] See [11].

[7] In [13].

[8] In [4], [9], and [17], respectively.

the CBD in ways roughly consistent with data from real cities. In some models, it can be established that an equilibrium density pattern is optimum. Under some restrictive conditions it can be shown that, if the population consists of two or more groups which differ only in exogenously determined incomes, then the higher income groups live further from the CBD and in lower density neighborhoods than the lower income groups, in both equilibrium and optimum solutions.[9]

An especially interesting aspect of the density issue concerns congestion in the transportation system. The conditions that make it optimum to use land near the CBD intensively for housing also make it optimum to use close-in land intensively for transportation. But an important way to use land intensively for transportation is by congestion. It is clearly optimum to have more transportation congestion close to the CBD than in the suburbs, and some interesting calculations have been made of the way that optimum congestion might vary with distance from the CBD. The new urban economics supports the presumption that congestion will be different from the optimum in equilibrium, but little is known about the details of possible comparisons.

Recent urban models have also shed important light on the process of urban decentralization or suburbanization. Many urban specialists attribute suburbanization to racial conflicts or to the desire by middle and upper income groups to avoid high central city taxes. Models that exclude these factors cannot of course prove that they are unimportant. But recent urban models conclude without exception that decentralization will certainly result from increases in income and decreases in the time and cost of commuting. Undoubtedly, models that located employment endogenously within the urban area would provide an even fuller explanation of suburbanization. In fact, the evidence is that decentralization pervades cities throughout the developed world. The surprising thing is that anyone would have attributed it mainly to the parochial problems of U. S. cities.

Perhaps the most intriguing result of the new urban economics is the finding that an urban structure that maximizes social welfare may require that people resident at different distances from the CBD achieve different utility levels. This may be so even in a model in which all citizens have the same utility functions, endowments, and skills, and in which the social welfare function is penalized by inequality. Nevertheless, social welfare maximization may require that per capita utility increase or decrease with distance from the CBD. The basic results appear in papers by Mirrlees, Dixit, and Riley.[10]

Perhaps the most important issue that has been attacked by the new urban economics is the ability of competitive markets to sustain an optimum urban structure. Some models have demonstrated circumstances under which the equilibrium urban structure is efficient. More important, some have studied market failure and public policies to correct resource misallocation. Most attention has been paid to market failure because of externalities resulting from high residential density or congestion in transportation. It is easy to un-

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[9] See, among others, Muth [13], and Montesano [12].

[10] See [11], [4], and [15], respectively.

derestimate the importance of this research unless one reads some of the literature that is written on urban problems without benefit of familiarity with welfare economics. City planning, the profession whose job is to improve on market resource allocation in urban areas, has been practically unaffected by welfare economics.

4. Future directions

■ Where will the new urban economics go in coming years? There are two fairly distinct reasons for studying urban models. The more obvious reason is to prepare for realistic models of urban growth and structure. It is often easier and cheaper to formulate, manipulate, and test relationships in models that are deliberately simplified and do not require much empirical input. The analogy in macroeconomics is that an important justification for much of macro theory is the construction of better Brookings models. At present the only available urban model in this category that is the work of economists is the NBER model.[11] The second reason for building theoretical urban models is to gain insight into, or to give an account of, a basic characteristic of urban areas by a model that contains a small number of relationships thought to be crucial to the characteristic in question. Examples of basic characteristics are the relative locations of different income groups, centrality, variable capital-land ratios, congestion, and decentralization. The new urban economics models mostly fall in this category. One analogy in macroeconomics is, of course, the outpouring of neoclassical growth models in the 1960s.

It is certainly desirable and inevitable for more models of the second kind to appear in coming years. There are several issues that can and should be studied in such models. Further work is needed to clarify the issue of optimum unequal utilities in urban areas. Does the finding of optimum inequality carry over to more complex models? If so, what can be said about the optimum amount and spatial distribution of inequality? It would be desirable to introduce population types that are diverse in skills or endowments of productive assets. What can be said about the optimum residential mixing of such diverse groups? It should also be possible to introduce diversity of tastes, e.g., for neighbors with similar race, religion, or income. Clearly, more attention is needed to the production side of urban structure. If there are two or more production sectors, what is their optimum spatial distribution, and can it be generated by competitive markets? Further study is also needed of the transportation sector. It should be possible to establish general conclusions about the urban size and structure that justify particular modal mixes.

Our hesitation about the future of the new urban economics stems from the fear that large amounts of technical expertise might be expended on relatively minor variations on a few themes. Especially with optimization models, working out even relatively minor variations can be a considerable technical chore. The analogy with growth theory is irresistible. Many economists feel that some of the neoclassical growth theory literature in the 1960s consisted of technical pyrotechnics that added neither insight nor realism to previous work. In fact, for almost every theoretical growth model there is an analogous urban model in which distance is substituted

[11] See [5].

for time. We are concerned that the profession will quickly see that Pontryagin's principle is as applicable to space as to time and that the journals will be flooded with technically sophisticated but economically uninteresting urban models. It is our judgment that much of the profession's effort at urban modeling should be devoted to making the models more realistic so that they can be used to plan public services, to evaluate private market performance, and to evaluate the desirability and efficacy of public regulation and control of private markets in urban areas.

We believe the profession faces an important issue in the method of formulating and analyzing more realistic models of urban structure. The rest of these notes present our views on this issue. The choice is between models in which space is treated as a continuous variable and models in which space is divided into a number of discrete sections. For convenience we refer to these as continuous and discrete models.

Continuous models are usually solved using variational methods or Pontryagin's principle. The advantage of these techniques is that, at least in fairly simple models, qualitative properties of the model can be derived. But in more complicated models a great deal of ad hoc numerical analysis is often required. A continuous model can be changed to a discrete model by dividing the land in the city into a finite number of sections, perhaps squares or circumferential rings. If the model contains n sections of land and m spatially undifferentiated goods, then the model is a general equilibrium system with $n + m$ prices. There are at least two different ways to formulate and solve such models. We discuss the two ways briefly and then indicate why we believe that discrete models are likely to prove more fruitful than continuous ones.

Economists have recently begun to solve nonlinear general equilibrium systems by using a class of fixed point algorithms first discovered by Herbert Scarf. Several of these algorithms and their applications to nonspatial equilibrium systems are described by Scarf and Hansen,[12] and a more effective algorithm is introduced by Kuhn and MacKinnon.[13] With a little ingenuity it is possible to solve discrete urban models by using these algorithms; some preliminary efforts to do so are discussed by MacKinnon.[14] The simplest approach is to divide the land outside the CBD into perhaps eight or ten rings, with all land in each ring assumed to be the same distance from the city center. Solution of the model means finding the equilibrium price of land in each ring. Since the solution process is almost entirely numerical, functional forms do not have to be chosen with so much care as they do for similar continuous models, and the model can be modified extensively without changing the method of solution. Unfortunately, this approach is only applicable when the object is to find an equilibrium, not when it is to find an optimum.

A promising approach to discrete optimizing urban models is to treat them as linear or nonlinear programming problems. Models of this type have been formulated by Mills.[15] Production is by standard programming technology in which input-output coefficients can vary

[12] In [16].
[13] In [6].
[14] In [7]
[15] In [9, 10].

with building height and, perhaps, with scale. Transportation takes place on a system whose capacity can be determined endogenously, and at costs that depend on congestion. The objective function is to minimize the cost of meeting stipulated goals, which might involve export quantities, consumption levels, etc. The location of production is determined endogenously; if there is a CBD, it comes out of the model rather than out of exogenous imposition. A bonus of the programming solution is that optimum prices emerge from the dual problem.

Both types of discrete model seem to offer sizeable advantages over continuous ones. One advantage is ease of use. To set up and solve a discrete model the first time usually involves considerable computer programming. But subsequently the model can be changed and re-solved at very small cost. Extra constraints can be added, functional forms can be changed, and, in the linear programming models, the number and type of sectors can be altered very easily, if the original programming has been done with foresight. To build a different continuous model, it is usually necessary to start from scratch again, deriving in most cases a new set of differential equations which have to be solved again by ad hoc numerical methods.[16] This is an important advantage, because investigating the effects of modifications is basic to much of the research in urban economics. How else can one study, for example, the effects of zoning regulations or racial segregation in housing markets in a general equilibrium context?

Using discrete models, researchers are less constrained than when using continuous ones to choose simple functional forms. For example, in his equilibrium model, Solow,[17] like many others, uses a logarithmic utility function, which implies that price and income elasticities of demand for housing are unity. He conjectures that the logarithmic form is not very limiting. We have no reason to doubt his judgment, but it is evidently difficult to find out just what difference other functional forms make. With the fixed point technique it should be possible to vary price and income elasticities routinely to determine their effects on the solution of the model. This is an important issue because the magnitudes of price and income elasticities of housing demand are major objects of current empirical research, and it is important to be able to incorporate the results of such research into urban models.

Similar remarks apply to other functions in urban models, such as housing production functions and functions representing transportation costs and technology. Discrete approaches permit the representation of a wide range of possible housing production technologies, whereas continuous approaches usually require that a simple production function, such as the Cobb-Douglas, be used. Likewise, continuous models require extremely simple transportation systems. In most, transportation costs can be monetary or psychic, but not both, and congestion cost functions are invariably extremely simple functions of the amount of congestion. Discrete models permit travel both to enter utility functions and to cost money. Speeds and costs may be varied from section to section and

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[16] See, for example, Solow [17, 19].

[17] In [17, 18].

in almost any way with congestion, and people can be permitted to choose among different modes of transportation.

It is somewhat ironic that economists are becoming proficient at building models in which all employment is in the CBD and all housing outside it, when cities look less and less like that paradigm. Major metropolitan areas have numerous nuclei of employment and commercial activity, not just a single CBD. Understanding the density and location of employment is at least as important as modeling the residential sector. Continuous models have so far had little to say on the subject. Discrete equilibrium models can at least examine the effects on residential location of multiple employment centers. And linear programming models make the location of productive activity completely endogenous.

If there is more than one employment nucleus, location cannot be indexed just by distance from the CBD. Space must be represented in two dimensions. This dramatically increases the complexity of continuous models. Equilibrium models would require the solution of systems of partial differential equations, and the mathematics of control theory has not even been worked out for this problem. For discrete problems, on the contrary, two dimensional space raises no problems of principle. If large cities or many housing or production sectors are involved, discrete models may, however, run into severe computational problems.

The chief disadvantage of discrete models is, of course, their almost complete dependence on numerical analysis and consequent lack of qualitative results. Moreover, a discrete theoretical city is only an approximation to a "true" continuous one, and this tends to offend one's theoretical sensibilities. No doubt an optimum research strategy would entail substantial efforts with both continuous and discrete approaches. There is much to be learned from both. The experience with growth theory in the 1960s leads us to suspect that the profession is not likely to underestimate the benefits from building continuous urban models. We fear that they may underestimate the benefits from building discrete ones.

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